Chapter XIII: Semantic Games for Fuzzy Logics

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1 Introduction

Deductive fuzzy logics nowadays come in many variants and types. Clearly there is no single logical system that is adequate for all applications and contexts. This fact imparts significance to the problem of *justifying particular logics* with respect to *specific principles of reasoning*. In other words, one is challenged to motivate the choice of a concrete logic with respect to basic models of reasoning that go beyond the mere presentation of some set of truth functions or of some proof system. Among the various models of fuzzy logics that have been proposed in this vein are Lawry's voting semantics [37], Paris's acceptability semantics [49], re-randomising semantics [34], and approximation semantics [6, 50]. This chapter addresses the challenge by presenting *semantic games* for some important fuzzy logics, in particular for Łukasiewicz logic, but also, e.g., for Product and for Gödel logic. Semantic games, sometimes also called evaluation games, characterize the evaluation of a given formula with respect to a given assignment of truth values to atomic formulas by a game between two players, that assume the roles of the proponent (or defender) and the opponent (or attacker) of the formula, respectively. This alternative to Tarski-style semantics was introduced by Jaako Hintikka in the late 1960s for classical logic [31]. Independently of Hintikka, Robin Giles suggested in the 1970s a game based interpretation of Łukasiewicz logic [25]. As will get clear in this chapter, both Hintikka's and Giles's games are starting points for a whole range of different game based characterizations of various important fuzzy logics.

A related enterprise, namely the connection between Ulam–Rényi games and t-norm based fuzzy logics, is presented in Chapter XIV of this volume of this Handbook. The reader should also be aware of the fact that there are many other kinds of logical games, that have at least partly been extended to many-valued logics as well: for example Lorenzen-style dialogue games, model comparison games (in particular Ehrenfreucht–Fraïssé games), and various forms of model construction games; however, here, we strictly focus on semantic games.

The chapter is structured as follows: We start by revisiting Hintikka's classical semantic game in Section 2 and observe that by simply placing this basic game into a many-valued setting one obtains a characterization of the so-called weak fragment of Łukasiewicz logic, also known as Kleene–Zadeh logic KZ. We also show that straightforward variations of Hintikka's game rules do not lead beyond connectives that are already definable in KZ. Section 3 presents a generalization of Hintikka's game to full Łukasiewicz logic, that requires the explicit reference to a truth value at any state of the game. In Section 4 we review Giles's game for Łukasiewicz logic, which is based on a more general concept of game states, but does not explicitly involve truth values during evaluation. Section 5 looks at Giles's game from a more abstract point of view and formulates dialogue as well payoff principles that should be maintained in generalizations and variants of that game if one aims at extracting truth functions from optimal strategies. We also show how games for other logics arise in this manner. Section 6 takes up a topic that has already been discussed by George Metcalfe in Chapter III (on proof theory) of this Handbook: the connection between Giles's game and logical rules for a particular type of hypersequent system. We present this concept from a somewhat different point of view here, and work out some details not for the original case of Łukasiewicz logic, but rather for Abelian logic instead. Section 7 describes another type of generalization of Hintikka's game that employs a stack of game states in addition to the current formula and role distribution. It is shown how Łukasiewicz, Gödel, as well as Product logic can be characterized in this manner. Section 8 presents yet another variant of semantic games based on introducing random choices, in addition to the choices made by the two players of a Hintikka-style game. In Section 9 the idea of considering random choices is lifted from the propositional level to rules for certain types of semi-fuzzy quantifiers in the context of a Giles-style game. Section 10 consists in a very concise synopsis of the various semantic games discussed in the previous sections. We close with historical remarks and hints on the sources in the final Section 11.

2 Hintikka's game — from classical to many-valued logics

2.1 Hintikka's classical semantic game

We will call Hintikka's classic game for characterizing truth in a given model the $\mathcal{H}$-game. Like in all semantic games that we will consider in this chapter, there are two players, called *myself* (or simply I) and *you*, here, who can both act either in the role of the *proponent* **P** or of the *opponent* **O**[1] of a given classical first-order formula φ. Throughout this chapter, instead of referring explicitly to variable assignments, we will assume that there is a constant in the language for each domain element of the interpretation with respect to which the formula is to be evaluated. For simplicity, we will identify the constant with the corresponding domain element. Initially I act as **P** and you act as **O**. My aim is to show that the initial formula is true in a given interpretation $\mathcal{M}$. More generally, in any state of the game, it is **P**'s aim to show that the formula in focus at the given state, called *current formula*, is true in $\mathcal{M}$. The game proceeds according to the following rules. Note that these rules only refer to the players' roles (*role distribution*) and to the outermost connective of the current formula.

($R^{\mathcal{H}}_{\wedge}$) If the current formula is $\varphi \wedge \psi$ then **O** chooses whether the game continues with φ or with ψ.

[1] See Section 11 for a remark on alternative names for roles and players.

$(R^{\mathcal{H}}_{\vee})$ If the current formula is $\varphi \vee \psi$ then **P** chooses whether the game continues with φ or with ψ.

$(R^{\mathcal{H}}_{\neg})$ If the current formula is $\neg\varphi$, the game continues with φ, except that the roles of the players are switched: the player who is currently acting as **P**, acts as **O** at the the next state, and vice versa for the current **O**.

$(R^{\mathcal{H}}_{\forall})$ If the current formula is $\forall x \varphi(x)$ then **O** chooses an element c of the domain of $\mathcal{M}$ and the game continues with $\varphi(c)$.

$(R^{\mathcal{H}}_{\exists})$ If the current formula is $\exists x \varphi(x)$ then **P** chooses an element c of the domain of $\mathcal{M}$ and the game continues with $\varphi(c)$.

Except for rule $R^{\mathcal{H}}_{\neg}$, the players' roles remain unchanged. The game ends when an atomic formula p is hit. The player who is currently acting as **P** *wins* and the other player, acting as **O**, *loses* if p is true in the given interpretation $\mathcal{M}$. We associate payoff 1 with winning and payoff 0 with losing. We also include the truth constants $\top$ and $\bot$, with their usual interpretation, among the atomic formulas. The game starting with formula φ is called the *$\mathcal{H}$-game for φ under $\mathcal{M}$*.

THEOREM 2.1.1 (Hintikka[2]). *I have a winning strategy in the $\mathcal{H}$-game for φ under the (classical) interpretation $\mathcal{M}$ iff φ is true in $\mathcal{M}$ (in symbols: $v_{\mathcal{M}}(\varphi) = 1$).*

2.2 Hintikka's game in a fuzzy logic setting

Recall that in game theory one is usually not just talking about winning or losing a game, but rather about the players' strategies for maximizing their payoffs. Since we have identified winning or losing the $\mathcal{H}$-game with receiving the payoff 0 or 1, respectively, this perspective also covers the $\mathcal{H}$-game: instead of talking about a winning strategy, we may refer to a strategy that guarantees payoff 1. With the exception of the explicit evaluation game in Section 3 we will employ this more general perspective here and identify truth values with payoff values that lie between 0 and 1. In this manner the $\mathcal{H}$-game can be straightforwardly generalized to a many-valued setting that is sometimes simply referred to as '(the) fuzzy logic', e.g. in the well known textbook [47]. This logic is occasionally also called the 'weak fragment of Łukasiewicz logic' or just 'weak Łukasiewicz logic' (see, e.g., [22]). Following [1], we prefer to call this logic *Kleene–Zadeh logic*, or KZ for short, here.

At the propositional level the semantics of KZ is specified by extending a given interpretation $\mathcal{M}$, i.e., an assignment $v_{\mathcal{M}}(\cdot)$ of propositional variables to truth values in $[0, 1]$, to arbitrary formulas as follows:

$$\begin{aligned} v_{\mathcal{M}}(\varphi \wedge \psi) &= \min\{v_{\mathcal{M}}(\varphi), v_{\mathcal{M}}(\psi)\} \\ v_{\mathcal{M}}(\varphi \vee \psi) &= \max\{v_{\mathcal{M}}(\varphi), v_{\mathcal{M}}(\psi)\} \\ v_{\mathcal{M}}(\neg\varphi) &= 1 - v_{\mathcal{M}}(\varphi) \\ v_{\mathcal{M}}(\bot) &= 0 \\ v_{\mathcal{M}}(\top) &= 1. \end{aligned}$$

[2] A proof for Theorem 2.1.1 can easily be extracted from the more general case of Theorem 2.2.2, below.

Interestingly, neither the rules nor the notion of a state in an $\mathcal{H}$-game have to be changed in order to characterize the logic KZ. We only have to generalize the possible payoff values for the $\mathcal{H}$-game from $\{0,1\}$ to the unit interval $[0,1]$, as already indicated above. More precisely, the payoff for the player who is in the role of **P** when a game under $\mathcal{M}$ ends with the atomic formula p is $v_{\mathcal{M}}(p)$, while the value for **O** is $1 - v_{\mathcal{M}}(p)$. If the payoffs are modified in this manner we will speak of an $\mathcal{H}$-mv-game *under* $\mathcal{M}$.

KZ can be straightforwardly extended to predicate logic. At the first-order level an interpretation $\mathcal{M}$ includes a non-empty set D as domain. With respect to our convention identifying domain elements with constants, the semantics of the universal and the existential quantifier is given by

$$v_{\mathcal{M}}(\forall x\varphi(x)) = \inf\{v_{\mathcal{M}}(\varphi(c)) \mid c \in D\}$$
$$v_{\mathcal{M}}(\exists x\varphi(x)) = \sup\{v_{\mathcal{M}}(\varphi(c)) \mid c \in D\}.$$

A slight complication arises for quantified formulas in $\mathcal{H}$-mv-games: there might be no element c in the domain of $\mathcal{M}$ such that $v_{\mathcal{M}}(\varphi(c)) = \inf\{v_{\mathcal{M}}(\varphi(c)) \mid c \in D\}$ or no domain element d such that $v_{\mathcal{M}}(\varphi(d)) = \sup\{v_{\mathcal{M}}(\varphi(d)) \mid d \in D\}$. A simple way to deal with this fact is to restrict attention to so-called witnessed models [29], where constants that witness all arising infima and suprema are assumed to exist. In other words: infima are minima and suprema are maxima in witnessed models. For KZ and for Łukasiewicz logic validity is not affected by restricting to witnessed models. However, for other logics like Gödel logic the set of formulas that always evaluate to 1 increases if only witnessed models are considered. In any case, we are not so much interested in validity here, but rather in concrete valuations. Therefore we adopt a more general solution that refers to optimal payoffs up to some ϵ.

DEFINITION 2.2.1. *Suppose that, for every $\epsilon > 0$, player **X** has a strategy that guarantees her a payoff of at least $w - \epsilon$, while her opponent has a strategy that ensures that **X**'s payoff is at most $w + \epsilon$, then w is called the* value for **X** of the game.

This notion, which corresponds to that of an ϵ-equilibrium as known from game theory, allows us to state the following generalization of Theorem 2.1.1.

THEOREM 2.2.2. *The value for myself of the $\mathcal{H}$-mv-game for φ under the interpretation $\mathcal{M}$ is w iff $v_{\mathcal{M}}(\varphi) = w$.*

Proof. We argue by induction on the complexity of φ. The induction hypothesis generalizes the statement in the theorem by referring to (the player in the role of) **P**, instead of just to myself, and by including that the value for **O** is $1 - v_{\mathcal{M}}(\varphi)$.

If φ is an atomic formula then the claim follows directly from Definition 2.2.1.

If $\varphi = \varphi_1 \vee \varphi_2$, then by rule $R^{\mathcal{H}}_{\vee}$ **P** can choose whether to continue the game with the current formula φ_1 or φ_2. By the induction hypothesis the value of the $\mathcal{H}$-mv-game for φ_i for the player in role **P** is $v_{\mathcal{M}}(\varphi_i)$ for $i = \{1,2\}$. To maximize her payoff **P** will therefore choose φ_i for $i \in \{1,2\}$ such that $v_{\mathcal{M}}(\varphi_i) = \max\{v_{\mathcal{M}}(\varphi_1), v_{\mathcal{M}}(\varphi_2)\}$. This guarantees that the value for the game for φ is $\max\{v_{\mathcal{M}}(\varphi_1), v_{\mathcal{M}}(\varphi_2)\} = v_{\mathcal{M}}(\varphi)$ as required. Moreover, it follows from the induction hypothesis that the value for **O**, after **P**'s choice of φ_i, is $\min\{1 - v_{\mathcal{M}}(\varphi_1), 1 - v_{\mathcal{M}}(\varphi_2)\} = 1 - \max\{v_{\mathcal{M}}(\varphi_1), v_{\mathcal{M}}(\varphi_2)\} = 1 - v_{\mathcal{M}}(\varphi)$, as required.

The case for $\varphi = \varphi_1 \wedge \varphi_2$ is like the one for $\varphi = \varphi_1 \vee \varphi_2$ with **P** and **O** switched and the case for negation follows directly from the induction hypothesis.

If $\varphi = \exists x \varphi'(x)$, then by the induction hypothesis and by Definition 2.2.1, we obtain that for every $\epsilon > 0$ player **P** has a strategy that guarantees her a payoff of at least $v_{\mathcal{M}}(\varphi'(c)) - \epsilon$ in the game for $\varphi'(c)$, for some domain element c. Therefore, for every $\delta > 0$ player **P** can pick a $d \in D$ such that her payoff in the game for $\varphi'(d)$ is not less than $\sup\{v_{\mathcal{M}}(\varphi'(d)) \mid d \in D\} - \delta = v_{\mathcal{M}}(\exists x \varphi'(x)) - \delta$. Analogously, we conclude that for each domain element d and for every $\delta > 0$ player **O** has a strategy to ensure that **P**'s payoff is at most $v_{\mathcal{M}}(\exists x \varphi'(x)) + \delta$ in a game for $\varphi'(d)$. Therefore the value for **P** of the game for $\varphi = \exists x \varphi'(x)$ is $\sup\{v_{\mathcal{M}}(\varphi'(d)) \mid d \in D\}$, as required. Taking into account that the value for **O** remains inverse to the one for **P**, we conclude that the induction hypothesis also holds for φ. □

2.3 Limits of Hintikka-style games

Note that there is no rule for implication in the $\mathcal{H}$-game or the $\mathcal{H}$-mv-game. Of course, we could simply define $\varphi \to \psi$ as $\neg\varphi \vee \psi$, like in classical logic. However this does not work for the (standard) implication of full Łukasiewicz logic Ł nor for other t-norm based fuzzy logics. As a consequence, at least at a first glimpse, the possibilities for extending $\mathcal{H}$-mv-games to logics more expressive than KZ look very limited if we insist on *Hintikka's principle* that a state of the game is fully determined by a formula and a distribution of the two roles (**P** and **O**) to the two players. There are only three elementary building blocks on which rules can be based: choices by **P**, choices by **O**, and role switch. By combining these building blocks one is led to a more general concept of propositional game rules, related to those described in [17] for connectives defined by arbitrary finite deterministic and non-deterministic matrices. In order to facilitate a concise specification of all rules of that type, we introduce the following technical notion.

DEFINITION 2.3.1. *An n-*selection *is a non-empty subset S of $\{1, \ldots, n\}$, where each element of S may additionally be marked by a* switch sign.

A game rule for an n-ary connective $\diamond$ in a *generalized $\mathcal{H}$-mv-game* is specified by a non-empty set $\{S_1, \ldots, S_m\}$ of n-selections. According to this concept, a round in a generalized $\mathcal{H}$-mv-game consists of two phases. The scheme for the corresponding game rule specified by $\{S_1, \ldots, S_m\}$ is as follows:

(Phase 1): If the current formula is $\diamond(\varphi_1, \ldots, \varphi_n)$ then **O** chooses an n-selection S_i from $\{S_1, \ldots, S_m\}$.

(Phase 2): **P** chooses an element $j \in S_i$. The game continues with formula φ_j, where the roles of the players are switched if j is marked by a switch sign.

REMARK 2.3.2. *A variant of this scheme arises by letting **P** choose the n-selection S_i in phase 1 and **O** choose $j \in S_i$ in phase 2. But note that playing the game for $\diamond(\varphi_1, \ldots, \varphi_n)$ according to that role inverted scheme is equivalent to playing the game for $\neg \diamond (\neg\varphi_1, \ldots, \neg\varphi_n)$ using the exhibited scheme.*

REMARK 2.3.3. *The rules $R^{\mathcal{H}}_{\wedge}$, $R^{\mathcal{H}}_{\vee}$, and $R^{\mathcal{H}}_{\neg}$ can be understood as instances of the above scheme:*
– *$R^{\mathcal{H}}_{\wedge}$ is specified by $\{\{1\},\{2\}\}$,*
– *$R^{\mathcal{H}}_{\vee}$ is specified by $\{\{1,2\}\}$, and*
– *$R^{\mathcal{H}}_{\neg}$ is specified by $\{\{1^*\}\}$, where the asterisk is used as switch mark.*

THEOREM 2.3.4. *In a generalized $\mathcal{H}$-mv-game, each rule of the type described above corresponds to a connective that is definable in the logic* KZ.

Proof. The argument for the adequateness of all semantic games considered in this paper proceeds by backward induction on the game tree.

For (generalized) $\mathcal{H}$-mv-games the base case is trivial: by definition **P** receives payoff $v_{\mathcal{M}}(p)$ and **O** receives payoff $1 - v_{\mathcal{M}}(p)$ if the game ends with the atomic formula p.

For the inductive case assume that the current formula is $\diamond(\varphi_1,\dots,\varphi_n)$ and that the rule for $\diamond$ is specified by the set $\{S_1,\dots,S_m\}$ of n-selections, where $S_i = \{j(i,1),\dots, j(i,k(i))\}$ for $1 \leq i \leq m$ and $1 \leq k(i) \leq n$. Remember that the elements of S_i are numbers $\in \{1,\dots,n\}$, possibly marked by a switch sign. For sake of clarity let us first assume that there are no switch signs, i.e., no role switches occur. Let us say that a player **X** *can force payoff* w if **X** has a strategy that guarantees her a payoff $\geq w$ at the end of the game. By the induction hypothesis, **P** can force payoff $v_{\mathcal{M}}(B)$ for herself and **O** can force payoff payoff $1 - v_{\mathcal{M}}(B)$ for himself if B is among $\{\varphi_1,\dots,\varphi_n\}$ and does indeed occur at a successor state to the current one; in other words, if $B = \varphi_{j(i,\ell)}$ for some $i \in \{1,\dots,m\}$ and $\ell \in \{1,\dots,k(i)\}$. Since **O** chooses the n-selection S_i, while **P** chooses an index number in S_i, **P** can force payoff

$$\min_{1\leq i\leq m}\ \max_{1\leq \ell\leq k(i)} v_{\mathcal{M}}(\varphi_{j(i,\ell)})$$

at the current state, while **O** can force payoff

$$\max_{1\leq i\leq m}\ \min_{1\leq \ell\leq k(i)} (1 - v_{\mathcal{M}}(\varphi_{j(i,\ell)})) = 1 - \min_{1\leq i\leq m}\ \max_{1\leq \ell\leq k(i)} v_{\mathcal{M}}(\varphi_{j(i,\ell)}).$$

If both players play optimally these payoff values are actually achieved. Therefore the upper expression corresponds to the truth function for $\diamond$. Both expressions have to be modified by uniformly substituting $1 - v_{\mathcal{M}}(\varphi_{j(i,\ell)})$ for $v_{\mathcal{M}}(\varphi_{j(i,\ell)})$ whenever $j(i,\ell)$ is marked by a switch sign in S_1 for $1 \leq i \leq m$ and $1 \leq k(i) \leq n$.

To infer that the connective $\diamond$ is definable in logic KZ it suffices to observe that its truth function, described above, can be composed from the functions $\lambda x(1-x)$, $\lambda x, y \min\{x,y\}$, and $\lambda x, y \max\{x,y\}$. But these functions are the truth functions for $\neg$, $\wedge$, and $\vee$, respectively, in KZ. □

Theorem 2.3.4 confirms our initial observation that there are hardly any options for generalizing game semantics to more expressive fuzzy logics, if we insist on Hintikka's principle that a game state should be fully determined by a formula and one of the two possible role distributions. We thus have to look for non-trivial augmentations of the $\mathcal{H}$-mv-game. A number of quite different such extensions will be considered in the succeeding sections.

3 An explicit evaluation game for Łukasiewicz logic

From the point of view of continuous t-norm based fuzzy logics, as popularized by Petr Hájek [28] and amply documented in this handbook, Kleene–Zadeh logic KZ is unsatisfying: while min is a t-norm, its residuum, which corresponds to implication in Gödel logic, is not expressible in KZ. If we define implication by $\varphi \to \psi =_{def} \neg\varphi \vee \psi$, in analogy to classical logic, then $\varphi \to \varphi$ is not valid, i.e., $v_{\mathcal{M}}(\varphi \to \varphi)$ is not true in all interpretations. In fact, formulas that do not contain truth constants are never valid in KZ.

The most important t-norm based fuzzy logic extending KZ is (full) Łukasiewicz logic Ł. The language of Ł extends that of KZ by implication $\to$, strong conjunction $\&$, and strong disjunction $\oplus$. The semantics of these connectives is given by

$$v_{\mathcal{M}}(\varphi \to \psi) = \min\{1, 1 - v_{\mathcal{M}}(\varphi) + v_{\mathcal{M}}(\psi)\}$$
$$v_{\mathcal{M}}(\varphi \,\&\, \psi) = \max\{0, v_{\mathcal{M}}(\varphi) + v_{\mathcal{M}}(\psi) -\})$$
$$v_{\mathcal{M}}(\varphi \oplus \psi) = \min\{1, v_{\mathcal{M}}(\varphi) + v_{\mathcal{M}}(\psi)\}.$$

All other propositional connectives could be defined in Ł, e.g., from $\to$ and $\bot$, or from $\&$ and $\neg$, alone. However, neither $\to$ nor $\&$ nor $\oplus$ can be defined in KZ. For this reason KZ is sometimes called the weak fragment of Ł, as mentioned in Section 2.

The increased expressiveness of Ł over KZ is particularly prominent at the first-order level: while in KZ there are only trivially valid formulas, which involve the truth constants in an essential manner, the set of valid first-order formulas in Ł is not even recursively enumerable, due to a classic result of Scarpellini [51].

As pointed out at the end of last section, it seems to be impossible to characterize full Łukasiewicz logic Ł by a trivial extension of the $\mathcal{H}$-game, comparable to the smooth shift from $\mathcal{H}$-games to $\mathcal{H}$-mv-games. However by introducing an explicit reference to a value $\in [0,1]$ at every state of the game we may define an *explicit evaluation game* or, shortly, $\mathcal{E}$-game for Ł.

Like above, we call the players *myself* (*I*) and *you*, and the roles **P** and **O**. In addition to the role distribution and the current formula, also a *current value* $\in [0,1]$ is included in the specification of a game state. We will thus denote $\mathcal{E}$-game states as pairs $\langle \varphi, r \rangle$, where φ is the current formula and r the current value. If $\langle \varphi, r \rangle$ is the initial state we speak of the $\mathcal{E}$-game *for* $\langle \varphi, r \rangle$.

Atomic formulas correspond to tests, like in the classical $\mathcal{H}$-game. If the current state is $\langle \alpha, r \rangle$, where α is an atomic formula, then the game ends and (the current) **P** wins if $v_{\mathcal{M}}(\alpha) \geq r$, otherwise **O** wins.

The rules for weak conjunction and disjunction remain essentially as in the $\mathcal{H}$-game, except for the additional reference to a value $r \in [0,1]$. This value however does not change in these moves.

($R^{\mathcal{E}}_{\wedge}$) If the current state $\langle \varphi \wedge \psi, r \rangle$ then **O** chooses whether the game continues with $\langle \varphi, r \rangle$ or with $\langle \psi, r \rangle$.

($R^{\mathcal{E}}_{\vee}$) If the current state is $\langle \varphi \vee \psi, r \rangle$ then **P** chooses whether the game continues with $\langle \varphi, r \rangle$ or with $\langle \psi, r \rangle$.

The rule for strong disjunction consists of two actions. First **P** divides the value of the current formula between the disjuncts; then **O** chooses one of the disjuncts (with the corresponding value) for the next state of the game.

($R^{\mathcal{E}}_{\oplus}$) If the current state is $\langle \varphi \oplus \psi, r\rangle$, then **P** chooses $r_\varphi, r_\psi \geq 0$ such that $r_\varphi + r_\psi = r$ and **O** chooses whether the game continues with $\langle \varphi, r_\varphi\rangle$ or with $\langle \psi, r_\psi\rangle$.

Note that the rule $R^{\mathcal{E}}_{\vee}$ for weak disjunction can be seen as a restricted case of the rule $R^{\mathcal{E}}_{\oplus}$ for strong disjunction, where either $r_\varphi = r$ and $r_\psi = 0$ or, conversely, $r_\psi = r$ and $r_\varphi = 0$.

Negation involves the switch of roles, as in the $\mathcal{H}$-game. However it is not sufficient to simply switch the roles **P** and **O** and to continue the game with the inverse value for the unnegated formula. The reason for this that we want to consistently interpret the statement to be defended by the player in role **P** at a state $\langle \psi, r\rangle$ as the claim that $v_{\mathcal{M}}(\psi) \geq r$. For $\psi = \neg\varphi$ this implies that the player has to defend $v_{\mathcal{M}}(\varphi) \leq 1 - r$ at the next state. After role switch the player who previously denied this claim acts in role **P** and therefore has to defend $v_{\mathcal{M}}(\varphi) > 1 - r$. To avoid the use of $>$ (or of $\leq$) instead of $\geq$, we reformulate the relevant condition as follows. If **O** denies **P**'s claim that $v_{\mathcal{M}}(\neg\varphi) \geq r$ then she asserts that $v_{\mathcal{M}}(\neg\varphi) < r$. This is equivalent to the claim that $v_{\mathcal{M}}(\neg\varphi) \leq r'$ for some r' strictly smaller than r, which in turn amounts to claiming $v_{\mathcal{M}}(\varphi) \geq 1 - r'$.

($R^{\mathcal{E}}_{\neg}$) If the current state is $\langle \neg\varphi, r\rangle$ then **O** chooses an r', where $0 \leq r' < r$, and the game continues with $\langle \varphi, 1 - r'\rangle$ with the roles of players switched.

The rule for the strong conjunction is dual to the one of strong disjunction. It again refers to two actions: modification of the value by **P** and a choice by **O**.

($R^{\mathcal{E}}_{\&}$) If the current state is $\langle \varphi \,\&\, \psi, r\rangle$ then **P** chooses a value $\bar{r}$, where $0 \leq \bar{r} \leq 1 - r$; then **O** chooses whether to continue the game with $\langle \varphi, r + \bar{r}\rangle$ or with $\langle \psi, 1 - \bar{r}\rangle$.

The universal quantifier rule is analogous to the one for the $\mathcal{H}$-game. The state $\langle \forall x \varphi(x), r\rangle$ corresponds to **P**'s claim that $\inf\{v_{\mathcal{M}}(\varphi(c)) \mid c \in D\} \geq r$. **O** has to provide a counterexample, i.e., to find a d such that $v_{\mathcal{M}}(\varphi(d)) < r$. Clearly the choice of a counterexample is independent of the (non)existence of an witnessing element for the infimum.

($R^{\mathcal{E}}_{\forall}$) If the current state is $\langle \forall x \varphi(x), r\rangle$ then **O** chooses some c in the domain D of the interpretation $\mathcal{M}$ and the game continues with $\langle \varphi(c), r\rangle$.

The situation is different for the existential quantifier. Now **P** has to provide a witness for the existential claim, i.e., for $\sup\{v_{\mathcal{M}}(\varphi(c)) \mid c \in D\} \geq r$. But as mentioned in Section 2, if the supremum is not a maximum, this poses a problem. It can happen that **P**'s claim is true, but that nevertheless there does not exist a witnessing element that would directly show this. The solution for the case of such non-witnessed models is similar to the one from Section 2. We relax the winning condition and allow the player who is currently in the role **P** to select a witness for which the value of the formula may not be equal to r, but is arbitrarily close. To this aim we let **O** decrease the value of the formula (where, of course, it is in **O**'s interest to decrease it as little as possible) and only *then* require **P** to find a witness (for the decreased value). Note that this does not affect **O**'s winning condition. If in the state $\langle \exists x\varphi(x), r\rangle$ the value r is strictly greater than $\sup\{v_{\mathcal{M}}(\varphi(c)) \mid c \in D\}$ then **O** can always win by choosing an ϵ that lies between the supremum and r. The just discussed rule can actually be stated formally without explicit reference to ϵ as follows.

($R^{\mathcal{E}}_{\exists}$) If the current state is $\langle \exists x\varphi(x), r\rangle$ then **O** chooses $r' < r$ and **P** chooses $c \in D$; the game continues with $\langle \varphi(c), \max\{0, r'\}\rangle$.

Compared to Theorems 2.1.1 and 2.2.2, the adequateness theorem for the $\mathcal{E}$-game reveals a somewhat less direct correspondence to the standard semantics of Ł.

THEOREM 3.0.1. *I, initially acting as **P**, have a winning strategy in the $\mathcal{E}$-game for $\langle \varphi, r\rangle$ under an Ł-interpretation $\mathcal{M}$ iff $v_{\mathcal{M}}(\varphi) \geq r$.*

Proof. We prove the claim by induction on complexity of φ. In the induction hypothesis we actually do not care whether I or you are initially in the role of **P**.

The base case, where χ is an atomic formula, is obvious.

If $\varphi = \varphi_1 \vee \varphi_2$ we argue similarly to the corresponding case of the proof of Theorem 2.2.2. By the induction hypothesis **P** has a winning strategy for $\langle \varphi_i, r_i\rangle$ iff $v_{\mathcal{M}}(\varphi_i) \geq r_i$ for $i \in \{1, 2\}$. **P**'s winning strategy for $\langle \varphi_1 \vee \varphi_2, r\rangle$ is obtained by the choice of φ_i such that $v_{\mathcal{M}}(\varphi_i) = \max\{v_{\mathcal{M}}(\varphi_1), v_{\mathcal{M}}(\varphi_2)\}$. Conversely a winning strategy for **P** for $\langle \varphi_1 \vee \varphi_2, r\rangle$ contains either one for $\langle \varphi_1, r\rangle$ or one for $\langle \varphi_2, r\rangle$. Therefore we obtain $\max\{v_{\mathcal{M}}(\varphi_1), v_{\mathcal{M}}(\varphi_2)\} \geq r$ by the induction hypothesis.

The cases for $\varphi = \varphi_1 \wedge \varphi_2$ is analogous.

Consider $\varphi = \neg\varphi_1$: we have $v_{\mathcal{M}}(\varphi) \geq r$ iff $v_{\mathcal{M}}(\varphi_i) \leq 1 - r$ iff $v_{\mathcal{M}}(\varphi_i) < 1 - r'$ for every $r' < r$. Now note that, since we are dealing with a finite game of perfect information, **O** has a winning strategy iff **P** does not have a winning strategy. Therefore the induction hypothesis implies that for every $r' < r$ **O** has a winning strategy in the game for $\langle \varphi_1, 1 - r'\rangle$ iff $v_{\mathcal{M}}(\neg\varphi_1) \geq r$. Since the rule $R^{\mathcal{E}}_{\neg}$ entails a role switch we obtain that **P** has a winning strategy for $\langle \varphi, r\rangle$ iff $v_{\mathcal{M}}(\varphi) \geq r$.

For $\varphi = \varphi_1 \oplus \varphi_2$ suppose that **P** has a winning strategy for $\langle \varphi, r\rangle$. By the rule $R^{\mathcal{E}}_{\oplus}$ this means that **P** has a winning strategy for $\langle \varphi_1, r_1\rangle$ as well as for $\langle \varphi_1, r_2\rangle$ for some r_1 and r_2 satisfying $r_1 + r_2 = r$. By the induction hypothesis we obtain that $v_{\mathcal{M}}(\varphi_1) \geq r_1$ and $v_{\mathcal{M}}(\varphi_2) \geq r_2$. But this implies that $v_{\mathcal{M}}(\varphi) = v_{\mathcal{M}}(\varphi_1 \oplus \varphi_2) = \min\{1, v_{\mathcal{M}}(\varphi_1) + v_{\mathcal{M}}(\varphi_2)\} \geq r_1 + r_2 = r$, as required. Conversely, suppose that $v_{\mathcal{M}}(\varphi) = v_{\mathcal{M}}(\varphi_1 \oplus \varphi_2) \geq r$. By the induction hypothesis **P** has a winning strategy for (φ_i, r_i) whenever $v_{\mathcal{M}}(\varphi_i) \geq r_i$, for $i \in \{1, 2\}$. Since $v_{\mathcal{M}}(\varphi_1 \oplus \varphi_2) = \min\{1, v_{\mathcal{M}}(\varphi_1) + v_{\mathcal{M}}(\varphi_2)\}$, **P** can choose $r_1 = v_{\mathcal{M}}(\varphi_1)$ and $r_2 = v_{\mathcal{M}}(\varphi_2)$ and combine the winning strategies for $\langle \varphi_1, r_1\rangle$ and $\langle \varphi_2, r_2\rangle$ into one for $\langle \varphi_1 \oplus \varphi_2, r_1 + r_2\rangle = \langle \varphi, r\rangle$.

For $\varphi = \varphi_1 \mathbin{\&} \varphi_2$ suppose that **P** has a winning strategy for $\langle \varphi, r\rangle$. By the rule $R^{\mathcal{E}}_{\&}$ this means that for some non-negative $\bar{r} \leq 1 - r$ **P** has a winning strategy for $\langle \varphi_1, r + \bar{r}\rangle$ as well as for $\langle \varphi_2, 1 - \bar{r}\rangle$. By the induction hypothesis we obtain that $v_{\mathcal{M}}(\varphi_1) \geq r + \bar{r}$ and $v_{\mathcal{M}}(\varphi_2) \geq 1 - \bar{r}$. Joining these facts we obtain that $v_{\mathcal{M}}(\varphi) = v_{\mathcal{M}}(\varphi_1 \mathbin{\&} \varphi_2) =$

$\max\{0, v_{\mathcal{M}}(\varphi_1)+v_{\mathcal{M}}(\varphi_2)-1\} \geq \max\{0, r+\bar{r}+1-\bar{r}-1\} = r$ as required. Conversely, suppose that $v_{\mathcal{M}}(\varphi) = v_{\mathcal{M}}(\varphi_1 \mathbin{\&} \varphi_2) \geq r$. Let **P** choose $\bar{r} = 1 - v_{\mathcal{M}}(\varphi_2)$. Then we have $v_{\mathcal{M}}(\varphi_2) = 1 - \bar{r}$. Because of $v_{\mathcal{M}}(\varphi_1 \mathbin{\&} \varphi_2) = \max\{0, v_{\mathcal{M}}(\varphi_1) + v_{\mathcal{M}}(\varphi_2) - 1\}$, we moreover have $v_{\mathcal{M}}(\varphi_1) = v_{\mathcal{M}}(\varphi_1 \mathbin{\&} \varphi_2) + 1 - v_{\mathcal{M}}(\varphi_2) = v_{\mathcal{M}}(\varphi_1 \mathbin{\&} \varphi_2) + \bar{r}$. Therefore, by the induction hypothesis, there exist winning strategies for **P** for the game $\langle \varphi_2, 1 - \bar{r} \rangle$ and for the game $\langle \varphi_1, r + \bar{r} \rangle$. By combining these strategies we obtain **P**'s winning strategy for $\langle \varphi, r \rangle$.

For $\varphi = \forall x \varphi_1(x)$ the rule $R^{\mathcal{E}}_{\forall}$ entails that **P** has a winning strategy for $\langle \varphi, r \rangle$ iff she has a winning strategy for $\langle \varphi_1(c), r \rangle$ for all $c \in D$. By the induction hypothesis the latter is equivalent to $v_{\mathcal{M}}(\varphi_1(c)) \geq r$ for all $c \in D$. But this in turn is equivalent to $v_{\mathcal{M}}(\forall x \varphi_1(x)) \geq r$.

For $\varphi = \exists x \varphi_1(x)$ let us once more check the two directions of the equivalence separately. First suppose that **P** has a winning strategy for $\langle \varphi, r \rangle$. By the rule $R^{\mathcal{E}}_{\exists}$ this means that for all $r' < r$ **P** can find a $c \in D$ such that she has a winning strategy for $\langle \varphi_1(c), \max\{0, r'\} \rangle$. By the induction hypothesis this implies $v_{\mathcal{M}}(\varphi_1(c)) \geq \max\{0, r'\}$ for all $r' < r$, and therefore $v_{\mathcal{M}}(\varphi) = v_{\mathcal{M}}(\exists x \varphi_1(x)) = \sup\{v_{\mathcal{M}}(\varphi_1(c)) \mid c \in D\} \geq \sup\{r' \mid r' < r\} = r$. Conversely, suppose $v_{\mathcal{M}}(\varphi) = v_{\mathcal{M}}(\exists x \varphi_1(x)) \geq r$. This implies that for every $r' < r$ there is $c \in D$ such that $v_{\mathcal{M}}(\varphi_1(c)) \geq r'$. By the induction hypothesis this implies that for every $r' < r$ there is a $c \in D$ such that **P** has a winning strategy for $\langle \varphi_1(c), r' \rangle$. According to rule $R^{\mathcal{E}}_{\exists}$ these winning strategies can be combined into one for $\langle \exists x \varphi_1(x), r \rangle = \langle \varphi, r \rangle$. $\square$

4 Giles's game for Łukasiewicz logic

Already in the 1970s Robin Giles [25] introduced a game that was intended to provide 'tangible meaning' to reasoning about statements with dispersive semantic tests as they appear in physics. For the logical rules of his game Giles referred not to Hintikka or Henkin, but rather to the dialogue game based semantics for intuitionistic logic by Lorenzen [39, 40]. In particular, Giles proposed the following rule for implication:

($R^{\mathcal{G}}_{\to}$) He who asserts $\varphi \to \psi$ agrees to assert ψ if his opponent will assert φ.

Like we did in Sections 2 and 3, above, Giles refers to the players as I and you, respectively. In contrast to $\mathcal{H}$-games, the rule $R^{\mathcal{G}}_{\to}$ introduces game states, where more than one formula may be currently asserted by each player. Since, in general, it matters whether we assert the same statement just once or more often, game states are now denoted as pairs of multisets of formulas. Following Giles, we call these multisets *my tenet* and *your tenet*, respectively. Formally we denote a state as

$$[\varphi_1, \ldots, \varphi_n \mid \psi_1, \ldots, \psi_m],$$

where $[\varphi_1, \ldots, \varphi_n]$ is your tenet and $[\psi_1, \ldots, \psi_m]$ is my tenet.

The payoff at a final game state, where all currently asserted formulas are atomic, is defined in terms of expected risks of payments to be made to the opposing player whenever an atomic assertion made by a player turns out to be false. More precisely, a *binary experiment* E_p is associated with each atomic formula p. 'Binary' here means that E_p either *fails* or *succeeds*. The special experiment $\mathsf{E}_\perp$ always fails. We stipulate that I have to pay 1€ to you for each of my assertions of E_p, where a corresponding trial of E_p fails. Likewise, you have to pay 1€ to me for each of your assertions that does not pass the associated test. The central feature of Giles's payoff scheme is that that each experiment E_p may be *dispersive*, meaning that E_p may yield different results when repeated. But a fixed *failure probability* $\pi(\mathsf{E}_p)$ is known to the players for each p; we call this probability the risk value $\langle p \rangle$ of p. Remember that it matters whether we assert the same proposition just once or more often. For a final game state

$$[p_1, \ldots, p_n \mid q_1, \ldots, q_m] \,.$$

we can therefore specify the expected total amount of money (in €) that I have to pay to you at the exhibited state by

$$\langle p_1, \ldots, p_n \mid q_1, \ldots, q_m \rangle = \sum_{1 \le i \le m} \langle q_i \rangle \;-\; \sum_{1 \le j \le n} \langle p_j \rangle \,.$$

We call this number briefly my *risk* associated with that state. Note that the risk can be negative, i.e., the risk values of the relevant propositions may be such that I expect an (average) net payment by you to myself.

As an example consider the state $[p, p \mid q]$, where you have asserted p twice and I have asserted q once. Three trials of experiments are involved in the corresponding evaluation: two trials of E_p, one for each of your assertions, and one trial of E_q to test my assertion. If $\langle p \rangle = 0.2$, i.e., if the probability that the experiment E_p fails is 0.2 and $\langle q \rangle = 0.5$ then $\langle p, p \mid q \rangle = 0.1$. This means that my expected loss of money according to the outlined betting scheme is 0.1€. On the other hand, if $\langle p \rangle = \langle q \rangle = 0.5$, then $\langle p, p \mid q \rangle = -0.5$, which means that I expect an (average) gain of 0.5€.

In the context of fuzzy logic, one may interpret this setup as a model of reasoning under vagueness. As linguists and philosophers of language have repeatedly pointed out, competent language users, in concrete dialogues, either (momentarily and provisionally) accept or don't accept utterances upon receiving them. No 'degrees of truth' enter the picture at this level; vagueness rather consists in a certain brittleness or dispersiveness of such highly context dependent decisions (see, e.g, [53]). One imagines that the dialogue partners repeatedly solicit answers to the question "Do you accept p?" from competent speakers who are familiar with the given context of assertion, but who may have different standards of acceptance of p, reflecting its vagueness. With respect to the terminology introduced above, the experiment E_p consists in asking this question; E_p fails if the answer is negative. To arrive at a 'degree of truth' for p we assume that the players have a particular expectation for E_p to fail or to succeed, that only depends on p. We may thus arrive at a many-valued interpretation by stipulating that $v_{\mathcal{M}}(p) = 1 - \langle p \rangle$.

Like in the $\mathcal{H}$-game and the $\mathcal{E}$-game, we distinguish the roles of a proponent **P** and of an opponent **O** for any occurrence of a complex formula. I act as **P** and you as **O** for any formula in my tenet, while you act as **P** and I act as **O** for any formula in your tenet. Following the terminology of Lorenzen for logical dialogue games, one also refers to a (semi-)move by a player in role **O** as an *attack* and calls the corresponding reaction of the other player (in role **P**) a *defense* of the attacked formula occurrence.

The rules $R^{\mathcal{H}}_{\wedge}$, $R^{\mathcal{H}}_{\vee}$, $R^{\mathcal{H}}_{\forall}$, and $R^{\mathcal{H}}_{\exists}$ defined in Section 2 basically remain unchanged for $\mathcal{G}$-games. However, we reformulate these rules as well as Giles's original implication rule, stated above, to better reflect the context in which these rules apply in a $\mathcal{G}$-game.

($R^{\mathcal{G}}_{\wedge}$) If the current formula is $\varphi \wedge \psi$ then the game continues in a state where the indicated occurrence of $\varphi \wedge \psi$ in **P**'s tenet is replaced by either φ or by ψ, according to **O**'s choice.

($R^{\mathcal{G}}_{\vee}$) If the current formula is $\varphi \vee \psi$ then the game continues in a state where the indicated occurrence of $\varphi \vee \psi$ in **P**'s tenet is replaced by either φ or by ψ, according to **P**'s choice.

($R^{\mathcal{G}}_{\rightarrow}$) If the current formula is $\varphi \rightarrow \psi$ then the indicated occurrence of $\varphi \rightarrow \psi$ is removed from **P**'s tenet and **O** chooses whether to continue the game at the resulting state or whether to add φ to **O**'s tenet and ψ to **P**'s tenet before continuing the game.

($R^{\mathcal{G}}_{\forall}$) If the current formula is $\forall x \varphi(x)$ then **O** chooses an element c of the domain of $\mathcal{M}$ and the game continues in a state where the indicated occurrence of $\forall x \varphi(x)$ in **P**'s tenet is replaced by $\varphi(c)$.

($R^{\mathcal{G}}_{\exists}$) If the current formula is $\exists x \varphi(x)$ then **P** chooses an element c of the domain of $\mathcal{M}$ and the game continues in a state where the indicated occurrence of $\exists x \varphi(x)$ in **P**'s tenet is replaced by $\varphi(c)$.

For later reference, we point out that $R^{\mathcal{G}}_{\rightarrow}$ contains a hidden *principle of limited liability*: referring to an occurrence of $\varphi \rightarrow \psi$, the player in role **O** may, instead of asserting φ in order to elicit **P**'s assertion of ψ, explicitly choose not to attack $\varphi \rightarrow \psi$ at all. This option results in a branching of the game tree. The state $[\Gamma \mid \Delta, \varphi \rightarrow \psi]$, where Γ and Δ are multisets of sentences asserted by you and me, respectively, and where the exhibited occurrence indicates that you currently refer to my assertion of $\varphi \rightarrow \psi$, has the two possible successor states $[\varphi, \Gamma \mid \Delta, \psi]$ and $[\Gamma \mid \Delta]$. In the latter state you have chosen to limit your liability in the following sense. Attacking an assertion by the other player should never incur an expected (positive) loss, which were the case if the risk associated with asserting φ is higher than that for asserting ψ. In such cases a rational player in role **O** will explicitly renounce an attack on $\varphi \rightarrow \psi$. For all other logical connectives the principle is ensured by the fact that the rules of the $\mathcal{G}$-game ensure that each occurrence of a formula can be attacked at most once: the attacked occurrence is removed from the state in the transition to a corresponding successor state.

Another form of the principle of limited liability arises for defense moves. In defending any sentence φ, **P** has to be able to hedge her possible loss associated with the assertions made in defense of φ to at most 1€. This is already the case for all logical rules considered so far. However, as shown in [14, 19], by making this principle explicit we arrive at a rule for strong conjunction, that is missing in Giles [25, 26]:

($R^{\mathcal{G}}_{\&}$) If the current formula is $\varphi \,\&\, \psi$ then **P** chooses whether to continue the game at a state where the indicated occurrence of $\varphi \,\&\, \psi$ is replaced by φ as well as ψ in **P**'s tenet, or by a single occurrence of $\bot$, instead.

The above description might yet be too informal to see in which sense every $\mathcal{G}$-game, just like an $\mathcal{H}$-game, constitutes an ordinary two-person zero-sum extensive game of finite depth with perfect information. For this purpose one should be a bit more precise than Giles and specify for each non-final state which player is to move next and which of the formulas in **P**'s tenet is the "current formula". For this purpose we introduce the notion of a *regulation*, which is a function ρ that maps every non-final game state $[\Gamma \mid \Delta]$ into an occurrence of some non-atomic formula in either your tenet Γ or in my tenet Δ; $\rho([\Gamma \mid \Delta])$ is called the *current formula*. When the current formula in a state has to be made explicit, we will underline it. If the initial state of a game is $[\;\mid \varphi]$ we speak of a *$\mathcal{G}$-game for φ*.

Given any Ł-interpretation $\mathcal{M}$ we define a corresponding risk value assignment by $\langle p \rangle_{\mathcal{M}} = 1 - v_{\mathcal{M}}(p)$ for every propositional variable p. Rather than to refer to the minimal upper bound of risks associated with the final states that I can enforce if we both play rationally, we want to talk about the *value* of a game, as defined in Section 2. To be able to apply Definition 2.2.1, we therefore stipulate that my risk at a final state (as defined above) is your payoff, while my payoff is the inverse of this risk, entailing that the game is zero-sum. These conventions allow us to formulate the characterization of Łukasiewicz logic Ł by $\mathcal{G}$-games as follows.

THEOREM 4.0.1. *The value for myself of a $\mathcal{G}$-game for φ under the risk value assignment $\langle\cdot\rangle_{\mathcal{M}}$ and an arbitrary regulation ρ is $v_{\mathcal{M}}(\varphi)$.*

Proof. Note that every run of a $\mathcal{G}$-game is finite; therefore we can once more 'solve' the game by backward induction. Recall that for every final state $[p_1, \ldots, p_m \mid q_1, \ldots, q_n]$, we have defined my associated risk as

$$\langle p_1, \ldots, p_n \mid q_1, \ldots, q_m \rangle = \sum_{1 \leq i \leq m} \langle q_i \rangle \; - \sum_{1 \leq j \leq n} \langle p_j \rangle \, .$$

To calculate the value of the game, i.e., the minimal upper bound of the final risk that I can enforce at a given non-final state S we have have to take into account two rationality principles, that arise since the $\mathcal{G}$-game is zero-sum:

1. If you can choose the successor state to S then my final risk at S is the maximum over all risks associated with successor states to S.

2. If, on the other hand, I can choose the successor state to S then my final risk at S is the minimum over all risks associated with the successor states.

Correspondingly, if the current formula is an implication, then the rule $R_{\rightarrow}^{\mathcal{G}}$ requires us to show that the notion of my risk $\langle \cdot \mid \cdot \rangle$ can be extended from final states to arbitrary states in a manner that guarantees that the following conditions are satisfied:

$$\left\langle \Gamma \mid \underline{\varphi \rightarrow \psi}, \Delta \right\rangle \;=\; \max\{\langle \Gamma \mid \Delta \rangle \, , \langle \Gamma, \varphi \mid \psi, \Delta \rangle\} \tag{1}$$

$$\left\langle \Gamma, \underline{\varphi \rightarrow \psi} \mid \Delta \right\rangle \;=\; \min\{\langle \Gamma \mid \Delta \rangle \, , \langle \Gamma, \psi \mid \varphi, \Delta \rangle\}. \tag{2}$$

To connect risk for arbitrary states with truth value assignments (Ł-interpretations) we extend the semantics of Ł from formulas to multisets Γ of formulas as follows:

$$v_{\mathcal{M}}(\Gamma) = \sum_{\varphi\in\Gamma} v_{\mathcal{M}}(\varphi).$$

Risk value assignments are in one to one correspondence with truth value assignments via $\langle p \rangle = \langle p \rangle_{\mathcal{M}} = 1 - v_{\mathcal{M}}(p)$ for all propositional variables p, which extends to

$$\langle p_1,\dots,p_m \mid q_1,\dots,q_n \rangle_{\mathcal{M}} = n - m + v_{\mathcal{M}}([p_1,\dots,p_m]) - v_{\mathcal{M}}([q_1,\dots,q_n]).$$

Correspondingly, we define the following function for arbitrary states:

$$\langle \Gamma \mid \Delta \rangle_{\mathcal{M}} = |\Delta| - |\Gamma| + v_{\mathcal{M}}(\Gamma) - v_{\mathcal{M}}(\Delta).$$

Note that this in particular entails

$$\langle \mid \varphi \rangle_{\mathcal{M}} = 1 - v_{\mathcal{M}}([\varphi]) = 1 - v_{\mathcal{M}}(\varphi). \tag{3}$$

It remains to show that $\langle \cdot \mid \cdot \rangle_{\mathcal{M}}$ indeed specifies my final risk at any state if I play rationally. For final states this is immediate. If the current formula selected by the regulation ρ is an implication in my tenet then we have to check that $\langle \cdot \mid \cdot \rangle_{\mathcal{M}}$ satisfies condition (1):

$$\begin{aligned}
\left\langle \Gamma \mid \underline{\varphi \to \psi}, \Delta \right\rangle_{\mathcal{M}} &= |\Delta| + 1 - |\Gamma| + v_{\mathcal{M}}(\Gamma) - v_{\mathcal{M}}(\Delta) - v_{\mathcal{M}}(\varphi \to \psi)\\
&= \langle \Gamma \mid \Delta \rangle_{\mathcal{M}} + 1 - v_{\mathcal{M}}(\varphi \to \psi)\\
&= \langle \Gamma \mid \Delta \rangle_{\mathcal{M}} + 1 - \min\{1, 1 - v_{\mathcal{M}}(\varphi) + v_{\mathcal{M}}(\psi)\}\\
&= \langle \Gamma \mid \Delta \rangle_{\mathcal{M}} - \min\{0, v_{\mathcal{M}}(\psi) - v_{\mathcal{M}}(\varphi)\}\\
&= \langle \Gamma \mid \Delta \rangle_{\mathcal{M}} + \max\{0, v_{\mathcal{M}}(\varphi) - v_{\mathcal{M}}(\psi)\}\\
&= \langle \Gamma \mid \Delta \rangle_{\mathcal{M}} + \max\{0, \langle \varphi \mid \psi \rangle_{\mathcal{M}}\}\\
&= \max\{\langle \Gamma \mid \Delta \rangle_{\mathcal{M}}, \langle \Gamma, \varphi \mid \psi, \Delta \rangle_{\mathcal{M}}\}.
\end{aligned}$$

For states where the current formula is an implication in your tenet condition (2) can be checked as follows:

$$\begin{aligned}
\left\langle \Gamma, \underline{\varphi \to \psi} \mid \Delta \right\rangle_{\mathcal{M}} &= |\Delta| - |\Gamma| - 1 + v_{\mathcal{M}}(\Gamma) + v_{\mathcal{M}}(\varphi \to \psi) - v_{\mathcal{M}}(\Delta)\\
&= \langle \Gamma \mid \Delta \rangle_{\mathcal{M}} - 1 + v_{\mathcal{M}}(\varphi \to \psi)\\
&= \langle \Gamma \mid \Delta \rangle_{\mathcal{M}} - 1 + \min\{1, 1 - v_{\mathcal{M}}(\varphi) + v_{\mathcal{M}}(\psi)\}\\
&= \langle \Gamma \mid \Delta \rangle_{\mathcal{M}} - 1 + \min\{1, 1 + \langle \psi \mid \varphi \rangle_{\mathcal{M}}\}\\
&= \langle \Gamma \mid \Delta \rangle_{\mathcal{M}} + \min\{0, \langle \psi \mid \varphi \rangle_{\mathcal{M}}\}\\
&= \min\{\langle \Gamma \mid \Delta \rangle_{\mathcal{M}}, \langle \Gamma, \psi \mid \varphi, \Delta \rangle_{\mathcal{M}}\}.
\end{aligned}$$

If the current formula is a strong conjunction, the following conditions arise from the rationality principles and the rule $R^{\mathcal{G}}_{\&}$:

$$\left\langle \Gamma \mid \underline{\varphi \,\&\, \psi}, \Delta \right\rangle = \min\{\langle \Gamma \mid \Delta, \bot \rangle, \langle \Gamma \mid \Delta, \varphi, \psi \rangle\} \tag{4}$$

$$\left\langle \Gamma, \underline{\varphi \,\&\, \psi} \mid \Delta \right\rangle = \max\{\langle \Gamma, \bot \mid \Delta \rangle, \langle \Gamma, \varphi, \psi \mid \Delta \rangle\}. \tag{5}$$

The corresponding arguments are as follows:

$$\begin{aligned}
\left\langle \Gamma \mid \underline{\varphi \,\&\, \psi}, \Delta \right\rangle_{\mathcal{M}} &= |\Delta| + 1 - |\Gamma| + v_{\mathcal{M}}(\Gamma) - v_{\mathcal{M}}(\Delta) - v_{\mathcal{M}}(\varphi \,\&\, \psi) \\
&= \langle \Gamma \mid \Delta \rangle_{\mathcal{M}} + 1 - v_{\mathcal{M}}(\varphi \,\&\, \psi) \\
&= \langle \Gamma \mid \Delta \rangle_{\mathcal{M}} + 1 - \max\{0, v_{\mathcal{M}}(\varphi) + v_{\mathcal{M}}(\psi) - 1\} \\
&= \langle \Gamma \mid \Delta \rangle_{\mathcal{M}} + \min\{1, (1 - v_{\mathcal{M}}(\varphi)) + (1 - v_{\mathcal{M}}(\psi))\} \\
&= \langle \Gamma \mid \Delta \rangle_{\mathcal{M}} + \min\{1, \langle \mid \varphi, \psi \rangle_{\mathcal{M}}\} \\
&= \min\{\langle \Gamma \mid \Delta, \bot \rangle_{\mathcal{M}}, \langle \Gamma \mid \Delta, \varphi, \psi \rangle_{\mathcal{M}}\}. \\
\left\langle \Gamma, \underline{\varphi \,\&\, \psi} \mid \Delta \right\rangle_{\mathcal{M}} &= |\Delta| - |\Gamma| - 1 + v_{\mathcal{M}}(\Gamma) + v_{\mathcal{M}}(\varphi \,\&\, \psi) - v_{\mathcal{M}}(\Delta) \\
&= \langle \Gamma \mid \Delta \rangle_{\mathcal{M}} - 1 + v_{\mathcal{M}}(\varphi \,\&\, \psi) \\
&= \langle \Gamma \mid \Delta \rangle_{\mathcal{M}} - 1 + \max\{0, v_{\mathcal{M}}(\varphi) + v_{\mathcal{M}}(\psi) - 1\} \\
&= \max\{\langle \Gamma, \bot \mid \Delta \rangle_{\mathcal{M}}, \langle \Gamma, \varphi, \psi \mid \Delta \rangle_{\mathcal{M}}\}.
\end{aligned}$$

Analogous conditions corresponding to the rules $R_{\vee}^{\mathcal{G}}$, $R_{\wedge}^{\mathcal{G}}$, $R_{\forall}^{\mathcal{G}}$, and $R_{\exists}^{\mathcal{G}}$, respectively, can be checked straightforwardly. But note that, just like in the corresponding cases of the $\mathcal{H}$-mv-game, the quantifier rules entail a reference to some 'margin of error' ϵ, as indicated in Definition 2.2.1 of the value of a game. □

REMARK 4.0.2. *Note that the above proof of Theorem 4.0.1 can be read as justification of Łukasiewicz logic with respect to Giles's game based model of approximate reasoning. Rather than imposing the truth functions for the various connectives in the first place, they are derived from the rules and the payoff scheme of the game in conjunction with the general concept of rationality that underlies game theory.*

At a first glimpse, Giles's game looks very different from the $\mathcal{H}$-mv-game. However, in a sense, it may actually be viewed as closer in spirit to the $\mathcal{H}$-mv-game (and therefore Hintikka's classic $\mathcal{H}$-game) than the $\mathcal{E}$-game described in Section 3. The main point here is that, in contrast to the $\mathcal{E}$-game, no explicit reference to truth values is made in the $\mathcal{G}$-game. Instead, like in the $\mathcal{H}$-mv-game, there is a direct match between (optimal) payoff for myself and truth values. Giles's motivation of payoff values in terms of bets on expected results of dispersive binary experiments seems to put his version of game semantics apart from Hintikka's. However note that the offered interpretation of truth values as inverted risk values is in fact completely independent from the semantic game itself. Therefore we may choose to ignore that part of Giles's semantic altogether and simply speak of assignments of values $\in [0, 1]$ to atomic formulas, just like for the $\mathcal{H}$-mv-game. Conversely, we may add Giles's betting scenario to the $\mathcal{H}$-mv-game and interpret the value assigned to an atomic formula as the inverted risk of having to pay 1€ when the claim that a particular dispersive experiment, characterized by a given failure probability, succeeds turns out to false. In other words, the only remaining essential difference between the $\mathcal{G}$-game and the $\mathcal{H}$-mv-game is that more than one formula occurrence has to be taken into account in general in the former case. In Sections 7 and 8 we will investigate variants of semantic games that address this issue. But before doing so, we will investigate (in Section 5) a generalization of the $\mathcal{G}$-game that leaves its notion of a game state unchanged and discuss (in Section 6) the relation of the $\mathcal{G}$-game and some of its variants to hypersequent based proof systems.

5 Generalizing Giles's game

In this section we review a general framework for Giles-style games at the propositional level. In contrast to the previous sections, we will not talk about specific rules for particular logical connectives, but rather specify a general rule format appropriate for this type of game. Moreover we will look at the evaluation of final (atomic) game states from a wider perspective that is neither dependent on particular motivations regarding reasoning in physics (as in Giles) nor on the presence of vagueness (as indicated in the Section 4). The main result recorded in this section is that truth functions over the reals—and in this sense: fuzzy logics—can be recovered for any concrete instance of this general game based framework.

We stick with the notion, introduced in Section 4, of a game state as consisting of two multisets of formulas: my tenet and your tenet. We will denote atomic tenets by γ or δ, possibly primed, and arbitrary tenets by upper Greek letters $\Gamma, \Delta, \ldots$. Moreover, we write $[\Gamma, \Delta]$ to denote the union of the multisets Γ and Δ as well as $[\Gamma, \phi]$ instead of $[\Gamma, [\phi]]$, etc.

5.1 General payoff principles

Giles's story about risking money to be paid when losing bets on dispersive experiments might be intriguing from a philosophical point of view, however, mathematically, it boils down to the definition of a particular ordinary payoff function in the usual game theoretic sense, i.e., an assignment of real numbers to all final states of the game. This observation motivates the formulation of general principles for assigning payoff values to atomic states. As in previous sections, we will only be interested in payoff for myself and thus simply speak of 'the payoff' associated with an atomic state. (More precisely, we can think of your payoff for the same state as directly inverse to mine. In other words, the game is zero-sum. This is codified in the *Payoff Principle* 2, below.

DEFINITION 5.1.1 (Payoff). *A* payoff function *assigns a real number to every atomic game state. The payoff of the atomic game state* $[\gamma \mid \delta]$ *is denoted as* $\langle \gamma \mid \delta \rangle$.

Payoff Principle 1 (Context independence)**.** *A payoff function* $\langle \cdot \mid \cdot \rangle$ *is* context independent *if for all atomic tenets* $\gamma, \delta, \gamma', \delta', \gamma''$*, and* δ'' *the following holds: If* $\langle \gamma' \mid \delta' \rangle = \langle \gamma'' \mid \delta'' \rangle$ *then* $\langle \gamma, \gamma' \mid \delta', \delta \rangle = \langle \gamma, \gamma'' \mid \delta'', \delta \rangle$.

Context independence entails that the payoff for a state $[\gamma, \gamma' \mid \delta, \delta']$ is solely determined by the payoffs of its sub-states $[\gamma \mid \delta]$ and $[\gamma' \mid \delta']$. This property is crucial for recovering a truth functional (compositional) semantics for all our games.

PROPOSITION 5.1.2. *Let* $\langle \cdot \mid \cdot \rangle$ *be a context independent payoff function and let* $G = [\gamma, \gamma' \mid \delta, \delta']$ *be an atomic game state. Then there exists an associative and commutative binary operation* $\oplus$ *on* $\mathbb{R}$ *such that* $\langle G \rangle = \langle \gamma \mid \delta \rangle \oplus \langle \gamma' \mid \delta' \rangle$.

Proof. Assume that $\langle \gamma \mid \delta \rangle = \langle \gamma'' \mid \delta'' \rangle = x$ and $\langle \gamma' \mid \delta' \rangle = \langle \gamma''' \mid \delta''' \rangle = y$. Then $\langle \gamma'', \gamma''' \mid \delta'', \delta''' \rangle = \langle \gamma, \gamma''' \mid \delta, \delta''' \rangle = \langle \gamma, \gamma' \mid \delta, \delta' \rangle$ by applying context independence twice. Thus we may write $\langle \gamma, \gamma' \mid \delta, \delta' \rangle = x \oplus y$. Associativity and commutativity of $\oplus$ directly follow from the fact that tenets are multisets. $\square$

REMARK 5.1.3. *We will call* $\oplus$ *as specified in Proposition 5.1.2 the* aggregation function corresponding to $\langle \cdot \mid \cdot \rangle$. *In Giles's original game the function* $\oplus$ *is ordinary addition, which motivates our notation.*

Payoff Principle 2 (Symmetry)**.** *A payoff function* $\langle \cdot \mid \cdot \rangle$ *is* symmetric *if* $\langle \gamma \mid \delta \rangle = -\langle \delta \mid \gamma \rangle$ *for all atomic tenets* γ *and* δ.

If $\langle \cdot \mid \cdot \rangle$ is context independent and symmetric then the payoff of an arbitrary atomic game state can be decomposed as follows:

$$\begin{aligned}\langle p_1, \dots, p_n \mid q_1, \dots, q_m \rangle &= \langle p_1 \mid \; \rangle \oplus \dots \oplus \langle p_n \mid \; \rangle \oplus \langle \; \mid q_1 \rangle \oplus \dots \oplus \langle \; \mid q_m \rangle \\ &= -\langle \; \mid p_1 \rangle \oplus \dots \oplus -\langle \; \mid p_n \rangle \oplus \langle \; \mid q_1 \rangle \oplus \dots \oplus \langle \; \mid q_m \rangle .\end{aligned}$$

Note that symmetry implies that $\langle \gamma \mid \gamma \rangle = 0$. In other words, the payoff is 0 in any atomic state where your tenet is identical to mine. Moreover, this shows that one could focus on single tenets instead of two-sided states.

PROPOSITION 5.1.4. *Let* $\langle \cdot \mid \cdot \rangle$ *be a context independent and symmetric payoff function. Then*

(i) $-$ *distributes over the corresponding aggregation function* $\oplus$, *i.e., for all payoff values* x *and* y, $-(x \oplus y) = -x \oplus -y$.

(ii) $-$ *is inverse to* $\oplus$, *i.e.,* $x \oplus -x = 0$ *holds for all values* x.

Proof. (i) Let $[\gamma_1 \mid \delta_1]$ and $[\gamma_2 \mid \delta_2]$ be two atomic states where $\langle \gamma_1 \mid \delta_1 \rangle = x$ and $\langle \gamma_2 \mid \delta_2 \rangle = y$. Then

$$\begin{aligned}-(x \oplus y) &= -(\langle \gamma_1 \mid \delta_1 \rangle \oplus \langle \gamma_1 \mid \delta_2 \rangle) && \text{by definition of } x, y \\ &= -\langle \gamma_1, \gamma_2 \mid \delta_1, \delta_2 \rangle && \text{by Proposition 5.1.2} \\ &= \langle \delta_1, \delta_2 \mid \gamma_1, \gamma_2 \rangle && \text{by Payoff Principle 1 (symmetry)} \\ &= \langle \delta_1 \mid \gamma_1 \rangle \oplus \langle \delta_2 \mid \gamma_2 \rangle && \text{by Proposition 5.1.2} \\ &= -\langle \gamma_1 \mid \delta_1 \rangle \oplus -\langle \gamma_2 \mid \delta_2 \rangle && \text{by Payoff Principle 1 (symmetry)} \\ &= -x \oplus -y && \text{by definition of } x, y.\end{aligned}$$

(ii) Let $[\gamma \mid \delta]$ be an atomic game state such that $\langle \gamma \mid \delta \rangle = x$. Then

$$\begin{aligned}x \oplus -x &= \langle \gamma \mid \delta \rangle \oplus -\langle \gamma \mid \delta \rangle && \text{by definition of } x \\ &= \langle \gamma \mid \delta \rangle \oplus \langle \delta \mid \gamma \rangle && \text{by Payoff Principle 1 (symmetry)} \\ &= \langle \gamma, \delta \mid \gamma, \delta \rangle && \text{by Proposition 5.1.2} \\ &= \langle \gamma \mid \gamma \rangle \oplus \langle \delta \mid \delta \rangle && \text{by Proposition 5.1.2} \\ &= 0 \oplus 0 && \text{by Payoff Principle 1 (symmetry)} \\ &= 0 && \text{by Proposition 5.1.2.}\end{aligned}$$

□

Note that every context independent and symmetric payoff function induces via its aggregation function a totally ordered Abelian group with (some subset of) the reals R as base set and with 0 as neutral element.

Given Proposition 5.1.4 we can rewrite the decomposition of the payoff for an atomic state $[p_1, \dots, p_n \mid q_1, \dots, q_m]$ as

$$\langle p_1, \dots, p_n \mid q_1, \dots, q_m \rangle = \bigoplus_{1 \le i \le m} \langle \; \mid q_i \rangle \oplus - \bigoplus_{1 \le j \le n} \langle \; \mid p_i \rangle .$$

Payoff Principle 3 (Monotonicity)**.** *A payoff function $\langle \cdot \mid \cdot \rangle$ is* monotone *if for all tenets γ,δ,γ', δ', γ'', and δ'' the following holds: if $\langle \gamma' \mid \delta' \rangle \leq \langle \gamma'' \mid \delta'' \rangle$ then $\langle \gamma, \gamma' \mid \delta', \delta \rangle \leq \langle \gamma, \gamma'' \mid \delta'', \delta \rangle$.*

PROPOSITION 5.1.5. *Let $\langle \cdot \mid \cdot \rangle$ be a monotone and context independent payoff function and $\oplus$ the corresponding aggregation function. Then for all payoff values x, y, z:*

(i) *If $y \leq z$, then $x \oplus y \leq x \oplus z$.*

(ii) $\min$ *and* $\max$ *distribute over $\oplus$, i.e.,* $\min\{x \oplus y, x \oplus z\} = x \oplus \min\{y, z\}$ *and* $\max\{x \oplus y, x \oplus z\} = x \oplus \max\{y, z\}$.

Proof. *(i)* Let $G = [\gamma \mid \delta]$, $\psi' = [\gamma' \mid \delta']$, and $\psi'' = [\gamma'' \mid \delta'']$ be three atomic states such that $\langle \psi \rangle = x, \langle \psi' \rangle = y$, and $\langle \psi'' \rangle = z$. Then the premise $y \leq z$ amounts to $\langle \gamma' \mid \delta' \rangle \leq \langle \gamma'' \mid \delta'' \rangle$ and $x \oplus y \leq x \oplus z$ to $\langle \gamma \mid \delta \rangle \oplus \langle \gamma' \mid \delta' \rangle \leq \langle \gamma \mid \delta \rangle \oplus \langle \gamma'' \mid \delta'' \rangle$ or, equivalently, to $\langle \gamma, \gamma' \mid \delta, \delta' \rangle \leq \langle \gamma, \gamma'' \mid \delta, \delta'' \rangle$, which is just an instance of Payoff Principle 3.

(ii) We only consider the equation for $\min$; the argument for $\max$ is analogous. Assume that $y \leq z$ holds. Then, by *(i)*, $x \oplus y \leq x \oplus z$ holds for all x and thus also $\min\{x \oplus y, x \oplus z\} = x \oplus y = x \oplus \min\{y, z\}$. On the other hand, if $z \leq y$ then $x \oplus z \leq x \oplus y$ and thus also $\min\{x \oplus y, x \oplus z\} = x \oplus z = x \oplus \min\{y, z\}$. ☐

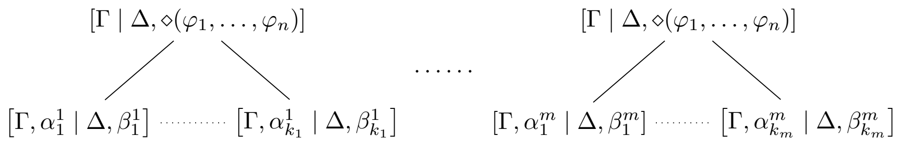
We combine the three payoff principles discussed in this section in the following notion, that will be central for Theorem 5.3.1 and Corollary 5.3.3 of Section 5.3.

DEFINITION 5.1.6 (Discriminating Payoff Function). *We call a payoff function $\langle \cdot \mid \cdot \rangle$* discriminating *if it is context independent, symmetric, and monotone.*

5.2 Dialogue principles for logical connectives

We now turn our attention to logical connectives and look for dialogue rules that regulate the stepwise reduction of states with logically complex assertions to final atomic states. We assume perfect information, which in particular implies that the two players have common knowledge of the payoff values. Since we strive for full generality, we will not consider conjunction, disjunction, implication, etc., separately, but rather specify a generic format of dialogue rules for arbitrary n-ary connectives ($n \geq 1$). It turns out that two simple and general *dialogue principles*, in combination with discriminating payoff functions, suffice to guarantee that a truth functional semantics can be extracted from the corresponding game.

Dialogue Principle 1 (Decomposition)**.** *A (dialogue) rule for an n-ary connective $\diamond$ is* decomposing *whenever in any corresponding round of the game exactly one occurrence of a compound formula $\diamond(\varphi_1, \ldots, \varphi_n)$ is removed from the current state and (possibly zero) occurrences of $\varphi_1, \ldots, \varphi_n$ and of truth constants are added to obtain the successor state. (Below we will give a step-by-step description of what is meant by 'round' here.)*



where α_j^i and β_j^i, for $1 \leq j \leq k_i$, $1 \leq i \leq m$, are multisets of zero or more occurrences of the formulas $\varphi_1, \ldots, \varphi_n$ and of truth constants.

Figure 1. Generic dialogue rule for your attack of my assertion of $\diamond(\varphi_1, \ldots, \varphi_n)$

Note that the decomposition principle entails that each occurrence of a formula can be attacked at most once: it is simply removed from the state in the corresponding round of the game. Moreover, an attack (i.e., a move by the player who is currently in the role of the opponent **O**) may or may not involve sub-formulas of the attacked formula occurrence (and/or truth constants) to be asserted by the attacking player. For example, remember that in Giles's $\mathcal{G}$-game, according to rule $R_{\rightarrow}^{\mathcal{G}}$, attacking $\varphi \rightarrow \psi$ requires the attacker to assert φ (see Section 4). We require the reply to any attack to follow at once. In our example of an attack to $\varphi \rightarrow \psi$ in the $\mathcal{G}$-game this means that an assertion of ψ will be added to the tenet of the attacked player. In general, the attacking player may choose between one of several available forms of attacking a particular formula, as witnessed by the rule for (weak) conjunction in the original game. Likewise, as exemplified in Giles's rule $R_{\vee}^{\mathcal{G}}$ for disjunction, a rule may also involve a choice on the side of the defending player. Consequently, every round of the game may be thought of as consisting of a sequence of three consecutive moves. (We only consider the case where you attack one of the formulas asserted by myself, the other case is dual.)

1. You pick an occurrence of a compound formula $\diamond(\psi_1, \ldots, \psi_n)$ from my current tenet for attack (or possibly for dismissal, see below).
2. You choose the form of attack (if there is more than one form available).
3. I choose the way in which I want to defend, i.e., to reply to the given attack on the indicated occurrence of $\diamond(\psi_1, \ldots \psi_n)$ (if such a choice is possible).

The corresponding rule may be depicted as shown in Figure 1. That there is a forest rather than a single tree rooted in $[\Gamma \mid \Delta, \diamond(\psi_1, \ldots, \psi_n)]$ reflects the fact that you may choose between different forms of attack for formulas of the form $\diamond(\psi_1, \ldots, \psi_n)$. In contrast, the branching in the trees corresponds to *my* possible choices in defending against your particular attack.

Recall from Section 4 that we have appealed to two forms of the *principle of limited liability* to explain the form of rules $R_{\rightarrow}^{\mathcal{G}}$ and $R_{\&}^{\mathcal{G}}$, respectively. In our current context it is appropriate to formulate this two-fold principle in the following more abstract manner:

Limited liability for defense (LLD): A player can always choose to assert $\bot$ in reply to an attack by her opponent.

Limited liability for attack (LLA): A player can always declare not to attack a particular occurrence of a formula that has been asserted by her opponent.

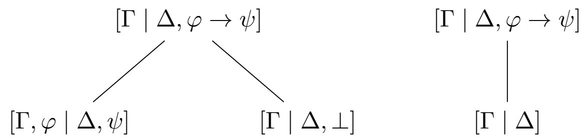


Figure 2. Implication rule (your attack) with two-fold principle of limited liability

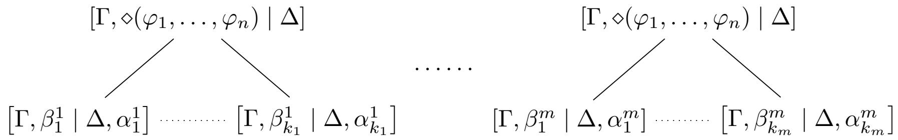


Figure 3. Generic dialogue rule dual to that in Figure 1; i.e., for my attack on your assertion of $\diamond(\varphi_1, \ldots, \varphi_n)$ (m, k_i, α^i_j, and β^i_j are like in Figure 1).

To illustrate the above dialogue rule format by a concrete example, consider the case of your attack on my assertion of $\varphi \to \psi$ in a variant of Giles's game where both forms of the principle of limited liability, LLD and LLA, are imposed. The resulting version of the implication rule is depicted in Figure 2.
The right (degenerate) tree in Figure 2 corresponds to your declaration not to attack the exhibited occurrence of $\varphi \to \psi$ at all. We treat this case as a special form of attack, where the 'attacked' formula occurrence (current formula) is simply removed to obtain the successor state. The first tree indicates a choice by me (i.e., the defending player): I may either according to LLD assert $\bot$ in reply to your attack or else assert ψ in exchange for your assertion of φ.

The second principle that we want to maintain in generalizing Giles's game is player neutrality, i.e., role duality: you and me have the very same obligations and rights in attacking or defending a particular type of formula.

Dialogue Principle 2 (Duality)**.** *Dialogue rule $\delta_\diamond$ for my (your) assertion of a formula of the form $\diamond(\varphi_1, \ldots, \varphi_n)$ is called* dual *to the rule $\delta'_\diamond$ for your (my) assertion of $\diamond(\varphi_1, \ldots, \varphi_n)$ if $\delta_\diamond$ is obtained from $\delta'_\diamond$ by just switching the roles of the players.*

We will say that a dialogue game has dual rules *whenever for every dialogue rule of the game there is a dual rule.*

Figure 3 depicts the generic dialogue that is dual to that in Figure 1. Note that now *I* am the one who, in attacking your assertion of $\diamond(\varphi_1, \ldots, \varphi_n)$, is free to pick a tree of the forest, whereas the branching in the tree now refers to *your* choices when defending against my attack.

Note that since the format of decomposing rules allows for a choice between different types of attacks as well as corresponding replies, we may speak without loss of generality of *the* dialogue rule for a connective $\diamond$ if the game has dual rules.

5.3 Extracting truth functions

Following the well known game theoretic principle of backward induction, that we have already seen at play in previous sections, the maximal payoff value that I can enforce at a game state S—for short: my *enforceable payoff* at S—amounts to the minimum of enforceable payoffs at the successor states of S if it is *your* turn to move at S as well as to the maximum of enforceable payoffs at the successor states if it is *my* turn to move at S. Correspondingly, the function $\langle \cdot \mid \cdot \rangle$ that denotes my enforceable payoff at an arbitrary state in our dialogue games (where a round involves a move by both of us in turn) is induced by the corresponding payoff function for atomic game states and by the following *min-max conditions* for non-atomic game states:

$$\langle \Gamma \mid \diamond(\varphi_1, \ldots \varphi_n), \Delta \rangle = \min_{1 \le i \le m} \max_{1 \le j \le k_i} \left\langle \Gamma, \alpha^i_j \mid \Delta, \beta^i_j \right\rangle \tag{6}$$

$$\langle \diamond(\varphi_1, \ldots \varphi_n), \Gamma \mid \Delta \rangle = \max_{1 \le i \le m} \min_{1 \le j \le k_i} \left\langle \Gamma, \beta^i_j \mid \Delta, \alpha^i_j \right\rangle, \tag{7}$$

where m, k_i, α^i_j, and β^i_j are defined as in Figure 1. We call this function the *extended payoff* function.[3]

Above, we have defined context independence, symmetry, and monotonicity for payoff functions which, by definition, refer only to atomic game states. However, by inspecting Definitions 1, 2, and 3 it is obvious that neither these properties, nor those expressed in Propositions 5.1.2, 5.1.4, and 5.1.5 depend on the atomicity of the formulas in a corresponding tenet. Therefore we can speak without ambiguity of context independence, symmetry, and monotonicity for arbitrary functions from general states to real numbers, not just for proper payoff functions.

THEOREM 5.3.1. *Let $\eth$ be a dialogue game with a discriminating payoff function and decomposing dual rules. Then the extended payoff function denoting my enforceable payoff is context independent, symmetric, and monotone.*

Proof. Given a discriminating payoff function $\langle \cdot \mid \cdot \rangle$ with corresponding aggregation function $\oplus$, we define a function v from (arbitrary) game states to the real numbers inductively as follows:

$$\begin{array}{ll} (a) & v([\ \mid p]) = \langle\ \mid p\rangle \\ (b) & v([\ \mid \Delta]) = \bigoplus_{\psi \in \Delta} v([\ \mid \psi])) \\ (c) & v([\Gamma \mid \Delta]) = v([\ \mid \Delta]) \oplus -v([\ \mid \Gamma]) \\ (d) & v([\ \mid \diamond(\varphi_1, \ldots \varphi_n)]) = \min\limits_{1 \le i \le m} \max\limits_{1 \le j \le k_i} v(\left[\alpha^i_j \mid \beta^i_j\right]), \end{array}$$

where m, k_i, α^i_j, and β^i_j are defined as in Figure 1.

[3] It can easily be checked that the above min-max conditions define a unique extension of any discriminating payoff function to arbitrary game states if the dialogue rules are dual and discriminating. As pointed out in [19] (for Giles's game) this fact implies that the order of rule applications is irrelevant: we arrive at the same enforceable payoff, independently of the specific formula occurrence that is picked by you or myself for attack at any given state.

We prove that v indeed calculates my enforceable payoff, i.e., it coincides with $\langle \cdot \mid \cdot \rangle$ on atomic states and fulfills the min-max conditions. Moreover we show that it is context independent, symmetric, and monotone.

It is straightforward to check that $v([\gamma \mid \delta])$ indeed coincides with $\langle \gamma \mid \delta \rangle$ for all atomic states $[\gamma \mid \delta]$. Following this observation we will from now on usually write $\langle \Gamma \mid \Delta \rangle$ instead of $v([\Gamma \mid \Delta])$, even if the tenets Γ and Δ are not atomic.

The symmetry of $v([\cdot \mid \cdot])$ immediately follows from its definition, where (here as well as further on) we freely exploit the commutativity and associativity of $\oplus$.

$$\begin{aligned} -v([\Gamma \mid \Delta]) = -\langle \Gamma \mid \Delta \rangle &= -(\langle \; \mid \Delta \rangle \oplus -\langle \; \mid \Gamma \rangle) && \text{by definition of } v \text{ (c)} \\ &= -\langle \; \mid \Delta \rangle \oplus \langle \; \mid \Gamma \rangle && \text{by Proposition 5.1.4(i)} \\ &= \langle \Delta \mid \Gamma \rangle && \text{by definition of } v \text{ (c).} \end{aligned}$$

Note that the definition of v directly entails that, just like the payoff at atomic states, the enforceable payoff at arbitrary states can also be obtained from the enforceable payoffs for sub-states by applying $\oplus$: we will refer to *merging* of and *partitioning*, respectively. More precisely:

$$\begin{aligned} \langle \Gamma, \Gamma' \mid \Delta', \Delta \rangle &= \langle \; \mid \Delta', \Delta \rangle \oplus -\langle \; \mid \Gamma, \Gamma' \rangle && \text{by definition of } v \text{ (c)} \\ &= (\langle \; \mid \Delta' \rangle \oplus \langle \; \mid \Delta \rangle) \oplus -(\langle \; \mid \Gamma' \rangle \oplus \langle \; \mid \Gamma \rangle) && \text{by definition of } v \text{ (b)} \\ &= \langle \; \mid \Delta' \rangle \oplus \langle \; \mid \Delta \rangle \oplus -\langle \; \mid \Gamma' \rangle \oplus -\langle \; \mid \Gamma \rangle && \text{by Proposition 5.1.4} \\ &= \langle \Gamma' \mid \Delta' \rangle \oplus \langle \Gamma \mid \Delta \rangle && \text{by definition of } v \text{ (c).} \end{aligned}$$

Given this fact, it is easy to see that $\langle \cdot \mid \cdot \rangle$ is context independent. Let $[\Gamma' \mid \Delta']$, $[\Gamma'' \mid \Delta'']$ be two game states such that $\langle \Gamma' \mid \Delta' \rangle = \langle \Gamma'' \mid \Delta'' \rangle$. Then for arbitrary tenets Γ and Δ:

$$\begin{aligned} \langle \Gamma, \Gamma' \mid \Delta', \Delta \rangle &= \langle \Gamma' \mid \Delta' \rangle \oplus \langle \Gamma \mid \Delta \rangle && \text{by partitioning} \\ &= \langle \Gamma'' \mid \Delta'' \rangle \oplus \langle \Gamma \mid \Delta \rangle && \text{by assumption} \\ &= \langle \Gamma, \Gamma'' \mid \Delta'', \Delta \rangle && \text{by merging.} \end{aligned}$$

Monotonicity also straightforwardly carries over from atomic to arbitrary game states. Let $[\Gamma' \mid \Delta']$, $[\Gamma'' \mid \Delta'']$ be two game states such that $\langle \Gamma' \mid \Delta' \rangle \leq \langle \Gamma'' \mid \Delta'' \rangle$. Then for arbitrary tenets Γ and Δ:

$$\begin{aligned} \langle \Gamma, \Gamma' \mid \Delta', \Delta \rangle &= \langle \Gamma' \mid \Delta' \rangle \oplus \langle \Gamma \mid \Delta \rangle && \text{by partitioning} \\ &\leq \langle \Gamma'' \mid \Delta'' \rangle \oplus \langle \Gamma \mid \Delta \rangle && \text{by assumption and Proposition 5.1.5(i)} \\ &= \langle \Gamma, \Gamma'' \mid \Delta, \Delta'' \rangle && \text{by merging.} \end{aligned}$$

It remains to check that the min-max conditions are satisfied. For states of the form $[\Gamma \mid \Delta, \diamond(\varphi_1, \ldots \varphi_n)]$ we obtain min-max condition (6) as follows:

$$\begin{aligned} &\langle \Gamma \mid \Delta, \diamond(\varphi_1, \ldots \varphi_n) \rangle \\ &\quad = \langle \Gamma \mid \Delta \rangle \oplus \langle \; \mid \diamond(\varphi_1, \ldots \varphi_n) \rangle && \text{by partitioning} \\ &\quad = \langle \Gamma \mid \Delta \rangle \oplus \min_{1 \leq i \leq m} \max_{1 \leq j \leq k_i} \left(\langle \alpha^i_j \mid \beta^i_j \rangle \right) && \text{by definition of } v \text{ (d)} \\ &\quad = \min_{1 \leq i \leq m} \max_{1 \leq j \leq k_i} \left(\langle \Gamma \mid \Delta \rangle \oplus \langle \alpha^i_j \mid \beta^i_j \rangle \right) && \text{by Proposition 5.1.5(ii)} \\ &\quad = \min_{1 \leq i \leq m} \max_{1 \leq j \leq k_i} \left(\langle \Gamma, \alpha^i_j \mid \beta^i_j, \Delta \rangle \right) && \text{by merging.} \end{aligned}$$

The dual min-max condition (7) exploits the symmetry of $\langle \cdot \mid \cdot \rangle$:

$$\begin{aligned}
&\langle \Gamma, \diamond(\varphi_1, \dots \varphi_n) \mid \Delta \rangle \\
&\qquad = -\langle \Delta \mid \Gamma, \diamond(\varphi_1, \dots \varphi_n) \rangle && \text{by symmetry} \\
&\qquad = -\min_{1 \leq i \leq m} \max_{1 \leq j \leq k_i} \left(\left\langle \Delta, \alpha^i_j \mid \beta^i_j, \Gamma \right\rangle \right) && \text{by min-max condition (6)} \\
&\qquad = \max_{1 \leq i \leq m} \min_{1 \leq j \leq k_i} \left(- \left\langle \Delta, \alpha^i_j \mid \beta^i_j, \Gamma \right\rangle \right) && \text{by Proposition 5.1.5(ii)} \\
&\qquad = \max_{1 \leq i \leq} \min_{1 \leq j \leq k_i} \left(\left\langle \beta^i_j, \Gamma \mid \Delta, \alpha^i_j \right\rangle \right) && \text{by symmetry,}
\end{aligned}$$

where m, k_i, α^i_j, and β^i_j are defined as in Figure 1. □

REMARK 5.3.2. *The duality of dialogue rules is used only indirectly in the above proof: it is reflected in the corresponding duality of the two min-max conditions and in the symmetry of the extended payoff function.*

COROLLARY 5.3.3. *Let* $\eth$ *be a game with discriminating payoff function and decomposing dual rules. Then for each connective* $\diamond$ *there is a function* $f_\diamond$ *such that* $\langle \mid \diamond(\varphi_1, \dots \varphi_n) \rangle = f_\diamond (\langle \mid \varphi_1 \rangle, \dots, \langle \mid \varphi_n \rangle)$ *for all formulas* $\varphi_1, \dots, \varphi_n$*, where* $\langle \cdot \mid \cdot \rangle$ *denotes the extended payoff function of Theorem 5.3.1.*

Proof. Applying min-max condition (6) as well as context independence and symmetry, we obtain

$$\begin{aligned}
\langle \mid \diamond(\varphi_1, \dots, \varphi_n) \rangle &= \min_{1 \leq i \leq m} \max_{1 \leq j \leq k_i} \left\langle \alpha^i_j \mid \beta^i_j \right\rangle \\
&= \min_{1 \leq i \leq m} \max_{1 \leq j \leq k_i} \left(\left\langle \mid \beta^i_j \right\rangle \oplus \left\langle \alpha^i_j \mid \right\rangle \right) \\
&= \min_{1 \leq i \leq m} \max_{1 \leq j \leq k_i} \left(\left\langle \mid \beta^i_j \right\rangle \oplus - \left\langle \mid \alpha^i_j \right\rangle \right) \\
&= \min_{1 \leq i \leq m} \max_{1 \leq j \leq k_i} \left(\bigoplus_{\beta \in \beta^i_j} \langle \mid \beta \rangle \oplus - \bigoplus_{\alpha \in \alpha^i_j} \langle \mid \alpha \rangle \right),
\end{aligned}$$

where $\oplus$ is the aggregation function corresponding to $\langle \cdot \mid \cdot \rangle$; m, k_i, β^i_j, and α^i_j obviously again refer to the dialogue rule for $\diamond(\varphi_1, \dots \varphi_n)$ as exhibited in Figure 1. Note that the α^i_js and β^i_js are multisets containing only the formulas $\varphi_1, \dots, \varphi_n$ and truth constants, which of course are evaluated to constant real numbers. Therefore that last expression defines the required function $f_\diamond$. □

REMARK 5.3.4. *To emphasize that* $f_\diamond$ *is of type* $\mathsf{R}^n \to \mathsf{R}$ *it can be rewritten as*

$$f_\diamond(x_1, \dots, x_m) = \min_{1 \leq i \leq n} \max_{1 \leq j \leq k_i} \left(\bigoplus_{y \in \overline{\beta^i_j}} y \oplus - \bigoplus_{x \in \overline{\alpha^i_j}} x \right),$$

where $\overline{\beta^i_j}$ *is a multiset of real numbers defined with respect to the multiset of formulas* β^i_j *as follows:* $\overline{\beta^i_j} = \{\overline{\varphi} \mid \varphi \in \beta^i_j\}$*, where* $\overline{\varphi} = x_i$ *when* $\varphi = \varphi_i$ *for* $1 \leq i \leq n$ *and* $\overline{\varphi} = \langle \mid \varphi \rangle$ *whenever* φ *is a truth constant.*

Note that the duality of the rules entails

$$\langle \diamond(\varphi_1, \dots \varphi_n) \mid \rangle = -\langle \mid \diamond(\varphi_1, \dots \varphi_n) \rangle = -f_\diamond (\langle \mid \varphi_1 \rangle, \dots, \langle \mid \varphi_n \rangle).$$

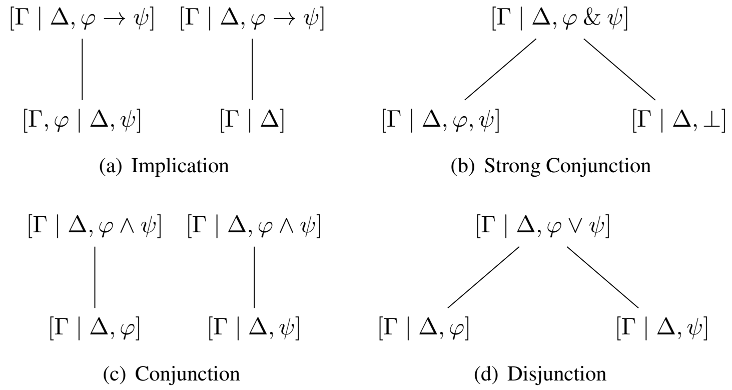


Figure 4. Rules of the $\mathcal{G}$-game for myself as **P** and you as **O**

By identifying payoff values with truth values we may thus claim to have extracted a unique truth function for $\diamond$ from a given payoff function and any decomposing dialogue rule for $\diamond$. However, as we will see in the next section, standard truth functions for many affected logics usually are based on different sets of truth values. To obtain those truth functions from an appropriate game we have to use certain bijections between payoff values and truth values, as we will explain below.

5.4 Revisiting the game for Łukasiewicz logic

To illustrate the emergence of concrete logics as instances of the general framework for games presented in Sections 5.1, 5.2, and 5.3, we should first check whether Giles's original $\mathcal{G}$-game for Łukasiewicz logic Ł is indeed covered. While the assignment of risk $\langle \cdot \mid \cdot \rangle$ to atomic states, as defined for the $\mathcal{G}$-game in Section 4, amounts to a discriminating payoff function (according to Definition 5.1.6), the connection to the standard truth functional semantics for Ł becomes clearer when we convert risk, that is to be minimized, to payoff, that is to be maximized, and set

$$\begin{aligned}
\langle p_1, \dots, p_n \mid q_1, \dots, q_m \rangle &= -\langle p_1, \dots, p_n \mid q_1, \dots, q_m \rangle \\
&= -\sum_{1 \le i \le m} \langle q_i \rangle + \sum_{1 \le j \le n} \langle p_j \rangle \\
&= -\sum_{1 \le i \le m} -\langle \mid q_i \rangle + \sum_{1 \le j \le n} -\langle \mid p_j \rangle \\
&= \sum_{1 \le i \le m} \langle \mid q_i \rangle - \sum_{1 \le j \le n} \langle \mid p_j \rangle \,.
\end{aligned}$$

Clearly, the aggregation function corresponding to $\langle \cdot \mid \cdot \rangle$ is ordinary addition. Figure 4 presents the dialogue rules in the format defined in Figures 1 and 3. Because of duality—which is obvious from Giles's generic presentation of the rules—we only have to consider your attacks on my assertions explicitly.

Note that discriminating payoff functions have 0 as neutral element. If we want to match the functions $f_{\rightarrow}$, $f_{\&}$, $f_{\wedge}$, and $f_{\vee}$ extracted from these dialogue rules according to Corollary 5.3.3 with standard truth functions over $[0, 1]$ we still have to add 1 to the payoff. It is straightforward to check that, modulo that transformation, the functions extracted from the rules in Figure 4 indeed coincide with the standard truth functions for Ł, reviewed in Section 4. We only illustrate the case for implication. From the rule for my assertion of $\varphi \rightarrow \psi$, which gives you a choice between asserting φ to force me to assert φ or else to declare that you will not attack this assertion at all, we obtain the following instance of min-max condition (1):

$$\langle \mid \varphi \rightarrow \psi \rangle = \min\{\langle \varphi \mid \psi \rangle, \langle \mid \rangle\} = \min\{0, \langle \mid \psi \rangle - \langle \mid \varphi \rangle\}.$$

Adding 1 yields the truth function $v(\varphi \rightarrow \psi) = 1 + \langle \mid \varphi \rightarrow \psi \rangle = \min\{1, 1 + \langle \mid \psi \rangle + 1 - (\langle \mid \varphi \rangle + 1)\} = \min\{1, 1 - v(\varphi) + v(\psi)\}$. The truth function for the other connectives are obtained in the same manner.

5.5 Finitely-valued Łukasiewicz logics

Instead of considering arbitrary risk (and therefore also arbitrary truth values) from $[0, 1]$, one may restrict the set of permissible risk values (equivalently: truth values) to $V_n = \{\frac{i}{n-1} \mid 1 \leq i < n\}$, for some $n \geq 2$. Since V_n is closed with respect to addition, subtraction, as well as $\min$ and $\max$, truth functions for all *finitely-valued* Łukasiewicz logics $Ł_n$ are obtained just like those for Ł.

Note that by this observation we have also covered classical logic, which coincides with $\mathbf{Ł}_2$. This means that classical logic can be modeled by a version of Giles's game where the experiments that determine the payoffs are not dispersive: every atomic proposition p is simply true or false, entailing a determinate payment of 1€ for every assertion of p in case it is false. For every assignment of risk values 0 or 1 to atomic formulas I have a strategy for avoiding (net) payment in a game starting with my assertion of a formula φ, if φ is true under that assignment; on the other hand, if φ is false, my best strategy limits my payment to you to 1€.

5.6 Cancellative hoop logic

A more interesting case is cancellative hoop logic CHL [10]. The truth value set of CHL is $(0, 1]$; correspondingly the truth constant $\bot$, along with negation $(\neg)$ is removed from the language. The truth functions for implication and strong conjunction are given as

$$\begin{aligned} v(\varphi \,\&\, \psi) &= v(\varphi) \cdot v(\psi) \\ v(\varphi \rightarrow \psi) &= \begin{cases} \frac{v(\psi)}{v(\varphi)} & \text{if } v(\varphi) > v(\psi) \\ 1 & \text{else.} \end{cases} \end{aligned}$$

At first sight it is unclear how to obtain these truth functions from dialogue rules in our framework. However remember that in the game for Łukasiewicz logics—assuming that Giles's "risk values" have already been translated into payoff values by multiplying with -1—we still had to shift payoff values by 1 to obtain the standard truth function $\tilde{\diamond}$ from the function $\varphi_{\diamond}$ that can be extracted from the dialogue rule for the connective $\diamond$. It will be helpful to visualize the general form of this relation, as follows:

$$\begin{array}{ccc} \mathcal{V}_{\text{payoff}} & \xrightarrow{f_\diamond} & \mathcal{V}_{\text{payoff}} \\ \mu\uparrow & & \downarrow\sigma \\ \mathcal{V}_{\text{truth}} & \xrightarrow{\tilde{\diamond}} & \mathcal{V}_{\text{truth}} \end{array}$$

In the case of Ł we have $\mathcal{V}_{\text{truth}} = [0,1]$, $\mathcal{V}_{\text{payoff}} = [-1,0]$, $\mu(x) = x - 1$, and $\sigma(x) = x + 1$. In CHL we have $\mathcal{V}_{\text{truth}} = (0,1]$. If we set $\mu(x) = \log(x)$ and accordingly $\mathcal{V}_{\text{payoff}} = (-\infty, 0]$ and $\rho(x) = \exp(x)$, then the implication rule of Giles's game (see Figure 4) yields the truth function for implication in CHL. In the same manner addition $(+)$ over $(-\infty, 0]$ maps into multiplication $(\cdot)$ over $(0,1]$. However, the function $f_\&$ extracted from the dialogue rule for $\&$ of Giles's game (with risk inverted into payoff) is $\&(x,y) = \max\{-1, x-1+y-1\}$ rather than the required $+$. (Note that the Łukasiewicz t-norm that models $\&$ in the standard semantics for Ł is obtained by adding $+1$, i.e., by applying σ, as explained above.) To obtain a dialogue rule for $\&$ such that $f_\& = +$, we have to drop the option to reply to an attack on $\varphi\&\psi$ by asserting $\bot$, instead of asserting φ and ψ. In other words we simply drop the principle of limited liability LLD from the original rule for strong conjunction.

5.7 Abelian logic

So far we have only considered logics where the set of truth values is a proper subset of R and where we had to explicitly transform payoff values into truth values and vice versa. But there is an interesting and well studied logic, namely Slaney and Meyer's Abelian logic A [23, 46] which coincides with one of Casari's logics for modeling comparative reasoning in natural language [7], where arbitrary real-valued payoffs in a Giles-style game can be directly interpreted as truth values. The truth value set of A indeed is R. The truth functions for implication ($\to$) is subtraction and the truth function for strong conjunction ($\&$) is addition over R. In addition, $\max$ and $\min$ serve as truth functions for disjunction ($\vee$) and weak conjunction ($\wedge$), respectively.

The game based characterization of A is particularly simple: just drop both forms of the principle of limited liability, LLA and LLD, from Giles's game. In other words: every assertion made by the opposing player, including those of the form $\varphi \to \psi$, has to be attacked, moreover the only permissible reply to attack an $\varphi\&\psi$ is to assert both φ and ψ. (The latter rule has already been used for CHL, above.) The functions that can be extracted from the resulting dialogue rules according to Corollary 5.3.3 are precisely those mentioned above: $f_\to = -$, $f_\& = +$, $f_\wedge = \min$, and $f_\vee = \max$.

We will revisit Abelian logic and present the corresponding game in greater detail in Section 6.

5.8 Alternative aggregation functions

In all the above examples, the aggregation function $\oplus$ corresponding to the respective payoff function has been addition $(+)$. This raises the question, whether in fact $\oplus$ always has to be $+$. This question is of some interest, since every truth function that can be directly extracted from a Giles-style game is built up from $\oplus$, $-$, $\min$, $\max$, and constant real numbers corresponding to truth constants. (By 'directly extracted' we mean:

disregarding further transformations—like $+1$ for Ł, and $\exp$ for CHL—that we may want to apply to map payoffs into standard truth values for particular logics.)

To settle this question in the negative it suffices to check that for any assignment v of reals to atomic propositions

$$\langle \gamma \mid \delta \rangle = \sqrt[3]{\sum_{q \in \delta} v(q)^3} - \sqrt[3]{\sum_{p \in \gamma} v(p)^3}$$

is a discriminating payoff function with $\oplus(x, y) = \sqrt[3]{x^3 + y^3}$ as corresponding aggregation function. However, we do not know of any many-valued logic in the literature where definitions of truth functions involve this or other possible aggregation functions different from $+$.

The above observations trigger the question whether for any aggregation function the ordered group $G = \langle \mathsf{R}; \leq, \oplus, 0, - \rangle$ is isomorphic to $\langle \mathsf{R}; \leq, +, 0, - \rangle$. A partly positive answer is provided by noting that ψ is Archimedean. This is essentially due to monotonicity (Payoff Principle 3) and the standard order $\leq$ on the base set R. Therefore Hölder's Theorem [36] entails that G is isomorphic to a *sub*group of $\langle \mathsf{R}; \leq, +, 0, - \rangle$.

6 Giles's game and hypersequents — the case of Abelian logic

Hypersequent systems are an important generalization of Gentzen's well known sequent framework for classical and intuitionistic logic. Roughly speaking, a hypersequent is a finite collection of sequents, viewed disjunctively. Chapter III of this Handbook not only demonstrates that hypersequents are a versatile tool for defining analytic proof systems for a wide variety of fuzzy logics, but also contains a section on Giles's game in this context. In particular, it is explained there in which sense the rules of the $\mathcal{G}$-game for Łukasiewicz logic Ł correspond to the logical rules of the hypersequent calculus GŁ. Obviously, this topic is also of central importance to the current handbook chapter. The relation between the $\mathcal{G}$-game and system GŁ is certainly the most important example of a direct connection between semantic games and cut-free inference systems for fuzzy logics; in fact, it is the only case of this kind that has been worked out in some detail in the literature so far. However, rather than repeating the material presented in Section 5.2 of [43], we want to address the issue from a slightly different angle and will work out the closely related case of Abelian logic A in some detail here.

We start with the Giles-style game for A that was briefly indicated in Section 5.7 and show that this semantic game (i.e., this game for *checking graded truth* in a given model) can be converted into a game for *checking validity* for formulas of A. The crucial step is to abstract away from concrete evaluations and to record the possible choices of the proponent **P** of the game in so-called *disjunctive states*. Different choices by the opponent **O** still correspond to different branches of the game tree, like in all games considered in the chapter. The only difference is that this game now consists in a tree of disjunctive states. At the leave nodes, where all formulas are atomic, we do not any longer just calculate my value or risk with respect to a particular interpretation, but rather have to check whether for *every interpretation* at least one of the disjunctive components of the final disjunctive state is a winning state in the sense of the original semantic game.

The just sketched disjunctive state scenario may be interpreted in two different manners: either as a new game in its own right, where the players have to move in ignorance of concrete risk value assignments (or corresponding interpretations), or, alternatively, simply as a device for *uniformly* computing winning strategies for **P** in the original semantic game. Here "uniformly" refers to the fact that we take into account all possible interpretations at every possible state of the game at once, rather than proceeding in a cases by case manner. (These two interpretations are explained in more detail at the end of Subsection 6.3.) Whatever interpretation one prefers, it turns out that at the level of strategies for disjunctive states (disjunctive strategies) the rules of the game for logic A directly correspond to the logical rules of a hypersequent system GA that is sound and cut-free complete for logic A.

6.1 Abelian logic revisited

The formulas of Abelian logic A are built up from propositional variables and the truth constant t using the connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$, and $\&$.[4] The semantics of A is often specified with respect to arbitrary lattice ordered Abelian groups (Abelian ℓ-groups). However, in our context it is more appropriate to make use of the fact that it suffices to consider the particular ℓ-group $\langle \mathsf{R}, +, \max, -, 0\rangle$. More precisely, the set of real numbers R is taken as set of truth values and any corresponding interpretation $\mathcal{M}$ extends an assignment $v_\mathcal{M}$ of reals to propositional variables to arbitrary formulas as follows.

$$\begin{aligned}
v_\mathcal{M}(t) &= 0\\
v_\mathcal{M}(\varphi \vee \psi) &= \max\{v_\mathcal{M}(\varphi), v_\mathcal{M}(\psi)\}\\
v_\mathcal{M}(\varphi \wedge \psi) &= \min\{v_\mathcal{M}(\varphi), v_\mathcal{M}(\psi)\}\\
v_\mathcal{M}(\neg\varphi) &= -v_\mathcal{M}(\varphi)\\
v_\mathcal{M}(\varphi \rightarrow \psi) &= v_\mathcal{M}(\psi) - v_\mathcal{M}(\varphi)\\
v_\mathcal{M}(\varphi \,\&\, \psi) &= v_\mathcal{M}(\varphi) + v_\mathcal{M}(\psi).
\end{aligned}$$

A formula φ is called *valid* in A iff $v_\mathcal{M}(\varphi) \geq 0$ for every interpretation $\mathcal{M}$.

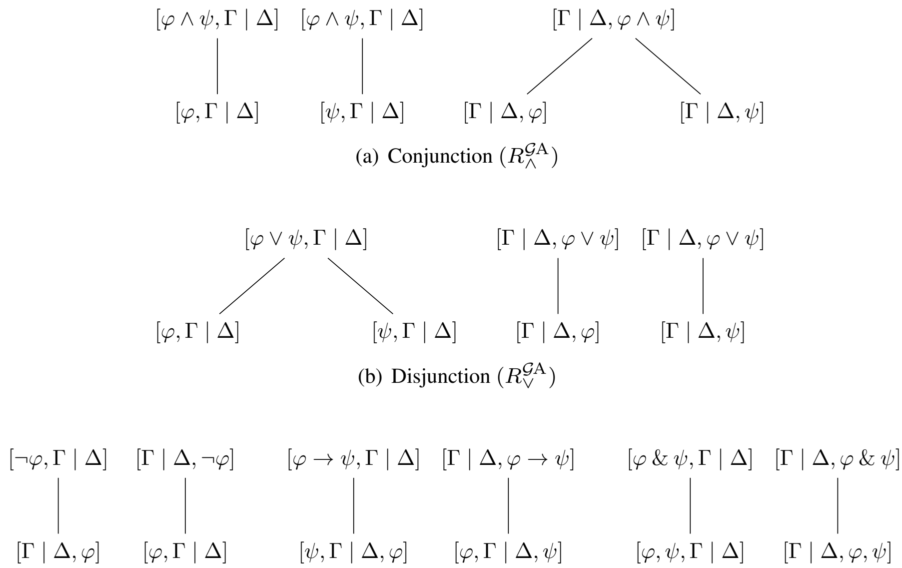
6.2 A Giles-style game for Abelian logic

We have already indicated in Section 5.7 how a semantic game can be obtained for logic A that is similar to Giles's $\mathcal{G}$-game for Ł. In particular, just like in the $\mathcal{G}$-game, each state in the corresponding $\mathcal{G}$A-game consists of a pair of multisets (tenets) of formulas denoted as

$$[\varphi_1, \ldots, \varphi_n \mid \psi_1, \ldots, \psi_m],$$

where $[\varphi_1, \ldots, \varphi_n]$ is your tenet and $[\psi_1, \ldots, \psi_m]$ is my tenet. Also recall that in every concrete instance of the game some *regulation* picks an occurrence of a non-atomic formula at any given state; with respect to this *current formula*, I act as proponent **P** and you act as opponent **O** if the occurrence is in my tenet. The roles are switched if the current formula is in your tenet. We state the rules of the $\mathcal{G}$A-game explicitly; in the notation introduced in Section 5, we can depict these rules as in Figure 5:

[4] Strong conjunction ($\&$) for Abelian logic is often denoted by $+$ and negation $\neg$ as $-$. Moreover 0 and e are alternative signs for the truth constant denoted by t here.



(a) Conjunction ($R_{\wedge}^{\mathcal{G}\mathrm{A}}$)

(b) Disjunction ($R_{\vee}^{\mathcal{G}\mathrm{A}}$)

(c) Negation ($R_{\neg}^{\mathcal{G}\mathrm{A}}$) (d) Implication ($R_{\to}^{\mathcal{G}\mathrm{A}}$) (e) Strong Conjunction ($R_{\&}^{\mathcal{G}\mathrm{A}}$)

Figure 5. Rules of the $\mathcal{G}$A-game

($R_{\wedge}^{\mathcal{G}\mathrm{A}}$) If the current formula is $\varphi \wedge \psi$ then the game continues in a state where the indicated occurrence of $\varphi \wedge \psi$ in **P**'s tenet is replaced by either φ or by ψ, according to **O**'s choice.

($R_{\vee}^{\mathcal{G}\mathrm{A}}$) If the current formula is $\varphi \vee \psi$ then the game continues in a state where the indicated occurrence of $\varphi \vee \psi$ in **P**'s tenet is replaced by either φ or by ψ, according to **P**'s choice.

($R_{\neg}^{\mathcal{G}\mathrm{A}}$) If the current formula is $\neg\varphi$ then the game continues in a state where the indicated occurrence of $\neg\varphi$ is removed from **P**'s tenet and an occurrence of φ is added to **O**'s tenet.

($R_{\to}^{\mathcal{G}\mathrm{A}}$) If the current formula is $\varphi \to \psi$ then the game continues in a state where the indicated occurrence of $\varphi \to \psi$ is replaced by ψ in **P**'s tenet and an occurrence of φ is added to **O**'s tenet.

($R_{\&}^{\mathcal{G}\mathrm{A}}$) If the current formula is $\varphi \,\&\, \psi$ then the game continues in a state where the indicated occurrence of $\varphi \,\&\, \psi$ in **P**'s tenet is removed and an occurrence of φ as well as an occurrence of ψ are added to **P**'s tenet.

REMARK 6.2.1. *It is instructive to compare the rules of the $\mathcal{G}\mathrm{A}$-game with those of the $\mathcal{G}$-game. The $\mathcal{G}\mathrm{A}$-game-rules for $\wedge$ and $\vee$ are exactly as in the $\mathcal{G}$-game, and therefore basically like already in Hintikka's $\mathcal{H}$-game. Rule $R_{\rightarrow}^{\mathcal{G}\mathrm{A}}$ differs from the implication rule $R_{\rightarrow}^{\mathcal{G}}$ of the $\mathcal{G}$-game, since the latter rule gives **O** the option not to attack **P**'s assertion of the current formula and consequently to have it removed without replacement from **P**'s tenet. This option, which is an instance of the principle of limited liability for attack (LLA), referred to in Sections 4 and 5, is missing in $R_{\rightarrow}^{\mathcal{G}\mathrm{A}}$: every occurrence of an implication has to be attacked. Similarly, rule $R_{\&}^{\mathcal{G}\mathrm{A}}$ differs from rule $R_{\&}^{\mathcal{G}}$, since the latter rule gives **P** the option to assert $\bot$ instead of the two conjuncts. In other words in the $\mathcal{G}$-game **P** can invoke the principle of limited liability for defense (LLD), whereas this option is missing in the $\mathcal{G}\mathrm{A}$-game. We did not formulate a negation rule for the $\mathcal{G}$-game, but rather pointed out that negation for Łukasiewic logic is defined by $\neg\varphi = \varphi \rightarrow \bot$. Actually, negation for Abelian logic can be defined analogously by $\neg\varphi = \varphi \rightarrow t$ and therefore we could also have omitted rule $R_{\neg}^{\mathcal{G}\mathrm{A}}$, in principle.*

At a final game state where $[p_1, \ldots, p_n]$ is your tenet and $[q_1, \ldots, q_m]$ is my tenet, the payoff for myself in a $\mathcal{G}\mathrm{A}$-game with respect to an interpretation $\mathcal{M}$ is specified as

$$\langle p_1, \ldots, p_n \mid q_1, \ldots, q_m \rangle = \sum_{1 \leq i \leq m} v_{\mathcal{M}}(q_i) - \sum_{1 \leq j \leq n} v_{\mathcal{M}}(p_j).$$

The following theorem follows from the general results of Section 5. It can also be shown directly in analogy to the proof of Theorem 4.0.1.

THEOREM 6.2.2. *In any $\mathcal{G}\mathrm{A}$-game for φ the value for myself under a given* A*-interpretation $\mathcal{M}$ and an arbitrary regulation ρ is $v_{\mathcal{M}}(\varphi)$.*

6.3 Disjunctive states

Note that Theorem 6.2.2 does not refer to validity in logic A, but rather to 'graded truth' in a given interpretation, like all other semantic games for fuzzy logics. However, given the definition of validity in A in Section 6.1 above, we obtain the following.

COROLLARY 6.3.1. *A formula φ is valid in* A *iff for every* A*-interpretation $\mathcal{M}$ and every regulation ρ the value for myself of the corresponding $\mathcal{G}\mathrm{A}$-game for φ is ≥ 0.*

My optimal strategies for a $\mathcal{G}\mathrm{A}$-game starting in state $[\mid \varphi]$, or in fact any non-final state for that matter, will of course depend on the given interpretation $\mathcal{M}$, in general. An inspection of the rules in Figure 5 reveals that there are only two cases where I have to make a choice: (1) if the current formula is an occurrence of $\varphi \vee \psi$ on my tenet then I, acting as **P**, have to decide whether to replace it by φ or by ψ; similarly, (2) if the current formula is an occurrence of $\varphi \wedge \psi$ on your tenet then I, acting as **O**, get to decide whether it should be replaced it by φ or by ψ. Recall that choices to be made by you amount to branching in any tree that represents a strategy for myself. To be able to keep track also of my own options we introduce the notion of a *disjunctive state*.[5] By this we just mean a finite multiset of ordinary states, written as

[5] Disjunctive states are also referred to as 'state disjunctions' (see [19, 43]). This notational ambiguity may be understood to reflect the two different interpretations of the corresponding rule system indicated at the end of this subsection.

$$\left[\varphi_1^1, \dots, \varphi_{n_1}^1 \mid \psi_1^1, \dots, \psi_{m_1}^1\right] \bigvee \dots \bigvee \left[\varphi_1^k, \dots, \varphi_{n_k}^k \mid \psi_1^k, \dots, \psi_{m_k}^k\right].$$

We now replace in part (a) of Figure 5

$$\begin{array}{cc} [\varphi \wedge \psi, \Gamma \mid \Delta] & [\varphi \wedge \psi, \Gamma \mid \Delta] \\ \Big| & \Big| \\ [\varphi, \Gamma \mid \Delta] & [\psi, \Gamma \mid \Delta] \end{array} \quad \text{by} \quad \begin{array}{c} [\varphi \wedge \psi, \Gamma \mid \Delta] \\ \Big| \\ [\varphi, \Gamma \mid \Delta] \bigvee [\psi, \Gamma \mid \Delta] \end{array}$$

and in part (b) of Figure 5

$$\begin{array}{cc} [\Gamma \mid \Delta, \varphi \vee \psi] & [\Gamma \mid \Delta, \varphi \vee \psi] \\ \Big| & \Big| \\ [\Gamma \mid \Delta, \varphi] & [\Gamma \mid \Delta, \psi] \end{array} \quad \text{by} \quad \begin{array}{c} [\Gamma \mid \Delta, \varphi \vee \psi] \\ \Big| \\ [\Gamma \mid \Delta, \varphi] \bigvee [\Gamma \mid \Delta, \psi]. \end{array}$$

We finally add an initial '$\mathcal{D} \bigvee$' to each state exhibited in Figure 5 and above, where $\mathcal{D}$ is a meta-variable for an arbitrary (possibly empty) disjunctive state. The resulting rule system can be interpreted in two different ways:

1. As a new game, where the states are now disjunctive states. For this new game we are not interested in payoff values at the final (disjunctive) states. Rather we declare that such a final disjunctive state, where all occurrences of formulas are atomic, is a *winning* state for myself iff for every interpretation $\mathcal{M}$ there is at least one disjunct $[p_1, \dots, p_n \mid q_1, \dots, q_m]$, such that $\langle p_1, \dots, p_n \mid q_1, \dots, q_m \rangle = \sum_{1 \le i \le m} v_{\mathcal{M}}(q_i) - \sum_{1 \le j \le n} v_{\mathcal{M}}(p_j) \ge 0$. Clearly, Corollary 6.3.1 implies that I have a winning strategy in this new game starting with the (single disjunct) state $[\ \mid \varphi]$ iff φ is valid in Abelian logic.

2. We may alternatively view the new rule system as a calculus for the systematic and uniform construction of optimal strategies for myself in the original $\mathcal{G}$A-game. To obtain such an optimal strategy for a given interpretation $\mathcal{M}$ one selects in each final disjunctive state a disjunct ($\mathcal{G}$A-game state) $[p_1, \dots, p_n \mid q_1, \dots, q_m]$, where $\langle p_1, \dots, p_n \mid q_1, \dots, q_m \rangle$ is maximal and removes all other disjuncts. Every remaining (non-disjunctive) final state can be traced to a unique parent $\mathcal{G}$A-game state (disjunct) in the parent node (disjunctive state) in the strategy tree. All disjuncts except this $\mathcal{G}$A-game state are removed from the parent state. This procedure is iterated until we arrive at an ordinary $\mathcal{G}$A-game strategy.

6.4 A hypersequent system for Abelian logic

As already mentioned at the beginning of this section, a hypersequent is a multiset of sequents, interpreted as disjunctions of sequents, and written as

$$\Gamma_1 \vdash \Delta_1 \mid \dots \mid \Gamma_n \vdash \Delta_n.$$

Sequents are understood here as pairs of multisets of formulas, just like states in Giles-style games. Consequently disjunctive states correspond to hypersequents.

Like an ordinary sequent calculus, hypersequent systems consist of axioms (initial hypersequents), structural rules, and logical rules. The following variant of a hypersequent system for Abelian logic can be found in [44].

Axioms:

$$(Ax) \quad \mathcal{G} \mid \Gamma, t^n \vdash \Gamma, t^m \quad \text{where } t^k \text{ denotes } k \geq 0 \text{ occurrences of } t$$

Structural rules:

$$\frac{\mathcal{G} \mid \Gamma \vdash \Delta \mid \Gamma \vdash \Delta}{\mathcal{G} \mid \Gamma \vdash \Delta}\ (EC) \qquad \frac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}{\mathcal{G} \mid \Gamma_1 \vdash \Delta_1 \mid \Gamma_2 \vdash \Delta_2}\ (\textit{Split})$$

Logical rules:

$$\frac{\mathcal{G} \mid \varphi, \Gamma \vdash \Delta \mid \psi, \Gamma \vdash \Delta}{\mathcal{G} \mid \varphi \wedge \psi, \Gamma \vdash \Delta}\ (\wedge, l) \qquad \frac{\mathcal{G} \mid \Gamma \vdash \Delta, \varphi \qquad \mathcal{G} \mid \Gamma \vdash \Delta, \psi}{\mathcal{G} \mid \Gamma \vdash \Delta, \varphi \wedge \psi}\ (\wedge, r)$$

$$\frac{\mathcal{G} \mid \varphi, \Gamma \vdash \Delta \qquad \mathcal{G} \mid \psi, \Gamma \vdash \Delta}{\mathcal{G} \mid \varphi \vee \psi, \Gamma \vdash \Delta}\ (\vee, l) \qquad \frac{\mathcal{G} \mid \Gamma \vdash \Delta, \varphi \mid \Gamma \vdash \Delta, \psi}{\mathcal{G} \mid \Gamma \vdash \Delta, \varphi \vee \psi}\ (\wedge, r)$$

$$\frac{\mathcal{G} \mid \Gamma \vdash \Delta, \varphi}{\mathcal{G} \mid \neg\varphi, \Gamma \vdash \Delta}\ (\neg, l) \qquad \frac{\mathcal{G} \mid \varphi, \Gamma \vdash \Delta}{\mathcal{G} \mid \Gamma \vdash \Delta, \neg\varphi}\ (\neg, r)$$

$$\frac{\mathcal{G} \mid \psi, \Gamma \vdash \Delta, \varphi}{\mathcal{G} \mid \varphi \to \psi, \Gamma \vdash \Delta}\ (\to, l) \qquad \frac{\mathcal{G} \mid \varphi, \Gamma \vdash \Delta, \psi}{\mathcal{G} \mid \Gamma \vdash \Delta, \varphi \to \psi}\ (\to, r)$$

$$\frac{\mathcal{G} \mid \varphi, \psi, \Gamma \vdash \Delta}{\mathcal{G} \mid \varphi \,\&\, \psi, \Gamma \vdash \Delta}\ (\&, l) \qquad \frac{\mathcal{G} \mid \Gamma \vdash \Delta, \varphi, \psi}{\mathcal{G} \mid \Gamma \vdash \Delta, \varphi \,\&\, \psi}\ (\&, r)$$

Note that the rule system for disjunctive states described in Section 6.3 can be obtained directly for the above logical rules by reading them bottom-to-top and replacing '$\mathcal{G}$' by '$\mathcal{D}$', '$|$' by '$\bigvee$', and '$\cdot \vdash \cdot$' by '$[\cdot \mid \cdot]$'. Conversely, we may view the logical hypersequent rules as directly derived from the rules for disjunctive game states.

Regarding the axioms and structural rules, it is important to realize that the hypersequent calculus remains complete for A if EC (external contraction) and *Split* are only applied to atomic hypersequents and if all instances of axioms are atomic. In fact this observation makes clear that the rule EC is redundant altogether. Since the truth constant t is interpreted by 0, every atomic instance of an axiom (Ax) satisfies the winning condition for the corresponding to final disjunctive game states. The rule *Split* has no direct correspondence in the rule system for disjunctive states. It may be seen as a device that allows one to reduce checking whether a given atomic hypersequent corresponds to a final disjunctive state satisfying the winning condition to the simpler case of Ax, where only the form of a single disjunct (sequent) is relevant.

Finally note that the particular order of applications of logical rules in a hypersequent derivation corresponds to a particular regulation in the game. Since every regulation leads to the same result (see Theorem 6.2.2 and Corollary 6.3.1), we conclude that systematic proof search in the hypersequent system is possible without backtracking at the level of logical rules.

7 Backtracking games

We have pointed out in Sections 3 and 4 that the $\mathcal{E}$-game as well as the $\mathcal{G}$-game deviate from what we called Hintikka's Principle, according to which a game state is fully determined by a single formula and a role distribution, i.e., the information whether myself or you are currently 'defending' the formula as the proponent **P**, while the other player is in the role of the opponent **O**. In this chapter we consider a variant of semantic games, where, unlike in the $\mathcal{E}$-game, no explicit reference to truth values is needed, and where, in contrast to the $\mathcal{G}$-game, the focus at each state is on a single formula. This is achieved by allowing the players to 'backtrack' to previous states of the game. Strictly speaking, the resulting *backtracking games* also do not satisfy Hintikka's Principle, since a stack of game states that are available for backtracking is now needed to fully describe a given state of a game. However the introduction of a stack for backtracking allows one to characterize not just Łukasiewicz logic Ł, but also Gödel logic G, and Product logic Π. In all previous sections it was always clear from the context to which logic we refer and thus we did not have to overload the notation for a corresponding evaluation function with an explicit reference to the logic in question. But in this section it is more appropriate to use $v^{Ł}_{\mathcal{M}}$, $v^{G}_{\mathcal{M}}$, and $v^{\Pi}_{\mathcal{M}}$ to refer the three corresponding valuation functions that specify the semantics of Ł, G, and Π, respectively. ($v^{Ł}_{\mathcal{M}}$ has been defined in Section 3. For the other two logics we will recall the relevant clauses for extending truth value assignments to compound formulas in the corresponding Subsections 7.2 and 7.4, respectively.)

REMARK 7.0.1. *In the following we will focus on the propositional level. However, we emphasize that adding Hintikka's original quantifier rules $R^{\mathcal{H}}_{\forall}$ and $R^{\mathcal{H}}_{\exists}$ results in characterizations of the corresponding first order logics, analogously to the case of the $\mathcal{H}$-mv-game for* KZ *and Giles's game for* Ł*.*

7.1 A backtracking game for Łukasiewicz logic

The backtracking game for Łukasiewicz logic may be viewed as a 'sequentialized' version of the $\mathcal{G}$-game. As announced, we introduce a *stack* on which information about an alternative state is stored (in a last-in first-out manner) when making particular moves. Initially the stack is empty. Upon reaching an atomic formula the game only ends if the stack is empty. Otherwise, the game *backtracks* to the state (formula and role distribution) indicated by the top element of the stack. That stack element is thereby removed from the stack.

In addition to the stack, we need to keep track of the *preliminary payoff* $\sigma_{\mathbf{P}}$ for the player that is currently acting as **P**. The preliminary payoff $\sigma_{\mathbf{O}}$ for **O** is $-\sigma_{\mathbf{P}}$ throughout the game. When the game ends, the preliminary payoff becomes final. Initially, $\sigma_{\mathbf{P}} = 1$. We call the resulting game variant the *backtracking game for* Ł or *$\mathcal{B}$Ł-game* for short.

The rules $R^{\mathcal{H}}_{\wedge}$, $R^{\mathcal{H}}_{\vee}$, and $R^{\mathcal{H}}_{\neg}$ of the $\mathcal{H}$-game of Section 2 are directly taken over into the $\mathcal{B}$Ł-game; no reference to the game stack or to $\sigma_{\mathbf{P}}$ and $\sigma_{\mathbf{O}}$ is needed. This implies that the $\mathcal{B}$Ł-game (for Ł) actually is an extension of the $\mathcal{H}$-mv-game for Kleene–Zadeh logic KZ (see Section 2). The rules for strong conjunction and implication are as follows ($\neg\varphi$ is treated as $\varphi \to \bot$):

$(R^{\mathcal{B}Ł}_{\&})$ If the current formula is $\varphi\&\psi$ then **P** can choose either (1) to continue the game with φ and to put ψ together with the current role distribution on the stack, or (2) to continue the game with $\bot$.

$(R^{\mathcal{B}Ł}_{\to})$ If the current formula is $\varphi \to \psi$ then **O** can choose either (1) to put ψ on the stack with the current role distribution and to continue the game with φ and inverted roles, or (2) to continue the game with the top element of the stack. If the stack is empty, the game ends.

$(R^{\mathcal{B}Ł}_{\text{at}})$ If the current formula is an atom p then $v_{\mathcal{M}}(p) - 1$ is added to $\sigma_{\mathbf{P}}$ and the same value is subtracted from $\sigma_{\mathbf{O}}$. The game ends if the stack is empty and is continued with the top element of the stack otherwise.

Again, we speak of the *$\mathcal{B}$Ł-game for φ under $\mathcal{M}$* if the game starts with the current formula φ where initially I am **P** and you are **O**.

THEOREM 7.1.1. *The value for myself of the $\mathcal{B}$Ł-game for φ under the Ł-interpretation $\mathcal{M}$ is w iff $v^{Ł}_{\mathcal{M}}(\varphi) = w$.*

Proof. We generalize to $\mathcal{B}$Ł-games that may start with any formula, role distribution, preliminary payoffs $\sigma_{\mathbf{P}} = -\sigma_{\mathbf{O}}$ and any stack content. We use $\mathcal{S}^I$ to denote the multiset of $|\mathcal{S}^I|$ formulas on the stack where I am assigned the role of **P**, and $\mathcal{S}^Y$ to denote the multiset of $|\mathcal{S}^Y|$ formulas on the stack where you are assigned the role of **P**. (Note that we ignore the order of stack elements, but not the number of occurrences of the same formula on the stack.) We define $s(\varphi) = 1$ if φ is atomic, $s(\neg\varphi) = s(\varphi) + 1$, and $s(\varphi \circ \varphi') = s(\varphi) + s(\varphi') + 1$ for $\circ \in \{\vee, \wedge, \&, \to\}$. We prove the following by induction on $n = s(\varphi) + \sum_{\psi \in \mathcal{S}^I \cup \mathcal{S}^Y} s(\psi)$: u is the value for myself of the $\mathcal{B}$Ł-game under interpretation $\mathcal{M}$ that starts with formula φ and with myself as **P** iff

$$u = \sigma_{\mathbf{P}} + v^{Ł}_{\mathcal{M}}(\varphi) - 1 + \sum_{\psi \in \mathcal{S}^I} (v^{Ł}_{\mathcal{M}}(\psi) - 1) - \sum_{\psi \in \mathcal{S}^Y} (v^{Ł}_{\mathcal{M}}(\psi) - 1).$$

The theorem follows for $\sigma_{\mathbf{P}} = 1$ and $|\mathcal{S}^I| = |\mathcal{S}^Y| = 0$. For the case where I am initially in the role of **O** we have

$$u = \sigma_{\mathbf{O}} - v^{Ł}_{\mathcal{M}}(\varphi) + 1 + \sum_{\psi \in \mathcal{S}^I} (v^{Ł}_{\mathcal{M}}(\psi) - 1) - \sum_{\psi \in \mathcal{S}^Y} (v^{Ł}_{\mathcal{M}}(\psi) - 1).$$

At the base case, $n = 1$, the stack is empty and φ is atomic. Therefore $v^{Ł}_{\mathcal{M}}(\varphi) - 1$ is added to $\sigma_{\mathbf{P}}$. The game ends at that state and $\sigma_{\mathbf{P}} + v^{Ł}_{\mathcal{M}}(\varphi) - 1$ is the payoff for myself as well as the value of the game, as required.

For the induction step we distinguish the following cases:

φ is atomic, but $n > 1$: $v^{Ł}_{\mathcal{M}}(\varphi) - 1$ is added to $\sigma_{\mathbf{P}}$ or is subtracted from $\sigma_{\mathbf{O}}$, depending on whether I am acting as **P** or as **O**. The game continues with the formula and role distribution at the top of the stack. Clearly, the induction hypothesis is preserved.

$\varphi = \varphi' \wedge \varphi''$: We continue either with the game where φ' or with the game where φ'' is the initial formula, according to **O**'s choice. Therefore we have to replace $v^{Ł}_{\mathcal{M}}(\varphi)$ by $\min\{v^{Ł}_{\mathcal{M}}(\varphi'), v^{Ł}_{\mathcal{M}}(\varphi'')\}$ to obtain the value for **P** of the original game from the values for **P** of the two possible succeeding games. This clearly matches the truth function for $\wedge$.

$\varphi = \varphi' \vee \varphi''$*:* like the case for $\varphi = \varphi' \wedge \varphi''$, except, since now **P** herself can choose the successor game, for replacing $v^{\text{Ł}}_{\mathcal{M}}(\varphi)$ by $\max\{v^{\text{Ł}}_{\mathcal{M}}(\varphi'), v^{\text{Ł}}_{\mathcal{M}}(\varphi'')\}$ in the value for **P**.

$\varphi = \varphi' \,\&\, \varphi''$*:* By the induction hypothesis, if $v^{\text{Ł}}_{\mathcal{M}}(\varphi') + v^{\text{Ł}}_{\mathcal{M}}(\varphi'') - 1$ is ≥ 0 then the value is maximized for **P** by choosing option (1): continue with φ', while putting φ'' on the stack. However, if $v^{\text{Ł}}_{\mathcal{M}}(\varphi') + v^{\text{Ł}}_{\mathcal{M}}(\varphi'') - 1$ is below 0 then **P** is better off by continuing the game with φ as new initial formula, i.e., choosing option (2) of rule $R^{\mathcal{B}\text{Ł}}_{\&}$. Therefore, putting the two options together, the value for **P** of the original game results from the values of the two possible succeeding games when we replace by $v^{\text{Ł}}_{\mathcal{M}}(\varphi)$ by $\max\{0, v^{\text{Ł}}_{\mathcal{M}}(\varphi') + v^{\text{Ł}}_{\mathcal{M}}(\varphi'') - 1\}$. This matches the truth function for $\&$.

$\varphi = \varphi' \to \varphi''$*:* If $v^{\text{Ł}}_{\mathcal{M}}(\varphi') > v^{\text{Ł}}_{\mathcal{M}}(\varphi'')$ then the (negative) contribution of φ' to the value of the game for **O** is higher than the (positive) contribution of φ'' for **O** and therefore **O** will choose option (1) of rule $R^{\mathcal{B}\text{Ł}}_{\to}$ and let the game continue with φ' and inverted roles, while φ'' is put on the stack. If, on the other hand, $v^{\text{Ł}}_{\mathcal{M}}(\varphi') \leq v^{\text{Ł}}_{\mathcal{M}}(\varphi'')$ then **O** will choose option (2) and discard φ altogether. In the latter case the game continues with the next formula/role distribution pair on the stack, unless the stack is empty and the game ends. Combining the two options we obtain the value for **P** from her values of the possible succeeding games as given by the induction hypothesis: we replace by $v^{\text{Ł}}_{\mathcal{M}}(\varphi)$ by $\min\{1, 1 - v^{\text{Ł}}_{\mathcal{M}}(\varphi'') + v^{\text{Ł}}_{\mathcal{M}}(\varphi')\}$. This matches the truth function for $\to$. □

REMARK 7.1.2. *An alternative way of proving Theorem 7.1.1 consists of transforming Giles's game into a $\mathcal{B}$Ł-game and vice versa.*

7.2 A backtracking game for Gödel logic

Like in KZ, but unlike in Ł, we only have to consider $\min$ and $\max$ as truth functions for conjunction and disjunction, respectively, for Gödel logic G. Recall that the semantics of implication in G is specified by $v^{\text{G}}_{\mathcal{M}}(\varphi \to \psi) = v^{\text{G}}_{\mathcal{M}}(\psi)$ if $v^{\text{G}}_{\mathcal{M}}(\varphi) > v^{\text{G}}_{\mathcal{M}}(\psi)$ and $v^{\text{G}}_{\mathcal{M}}(\varphi \to \psi) = 1$ otherwise. Negation is again defined by $\neg\varphi =_{def} \varphi \to \bot$ and therefore does not need separate consideration. To obtain a backtracking game for G, called $\mathcal{B}$G-game, we define the following rule:

($R^{\mathcal{B}\text{G}}_{\to}$) If the current formula is $\varphi \to \psi$ then the game is continued with ψ in the current role distribution and φ is put on the stack together with the inverse role distribution.

Note that no choice of the players is involved in this rule. Below, we will present an alternative implication rule with choice. Here, however, choices remain restricted to conjunctions and disjunctions, for which the rules $R^{\mathcal{H}}_{\wedge}$ and $R^{\mathcal{H}}_{\vee}$ of the $\mathcal{H}$-game remain in place.

($R^{\mathcal{B}\text{G}}_{\text{at}}$) If the current formula is atomic then the game ends if the stack is empty and is continued with the top element of the stack otherwise.

Keeping track of payoff values is more involved in the $\mathcal{B}$G-game than in the $\mathcal{B}$Ł-game. An (ordered) tree τ of all formula occurrences visited during the game is built up for that purpose. At a state where the current formula φ is a conjunction or a disjunction the subformula of φ chosen by **O** or **P**, respectively, is attached to τ as successor node to φ. If the current formula φ is an implication $\varphi' \to \varphi''$ then φ' and φ'' are attached to τ as the right and left successor node to φ, respectively. When an atomic formula p is reached then the corresponding leaf node p is labeled by $v^{\mathrm{G}}_{\mathcal{M}}(p)$. To compute the payoff at the end of a game, the values (labels) at the leaf nodes are finally propagated upwards in τ as follows. Let φ be the non-atomic formula at an internal node of τ, where each successor node has already been labeled by a value:

If $\varphi = \varphi' \to \varphi''$, then φ is labeled by 1 if $f' \leq f''$ and by f'' if $f' > f''$, where f' and f'' are the values that label φ' and φ'', respectively.

If $\varphi = \varphi' \vee \varphi''$ or $\varphi = \varphi' \wedge \varphi''$, then the same value that labels the successor node of φ also labels φ itself.

The $\mathcal{B}$G-*game for* φ *under* $\mathcal{M}$ starts with an empty stack, the current formula φ (that is also the initial tree τ) and the role distribution where I am **P** and you are **O**. The payoff for myself in that game is given by the label f of φ in τ (computed as explained above, once the game has ended). The payoff for you is $-f$. (In other words: the $\mathcal{B}$G-game is a zero sum game.)

THEOREM 7.2.1. *The value for myself of the* $\mathcal{B}$G-*game for* φ *under the* G-*interpretation* $\mathcal{M}$ *is* w *iff* $v^{\mathrm{G}}_{\mathcal{M}}(\varphi) = w$.

Proof. If φ does not contain $\wedge$ or $\vee$ then the tree τ of the game is just the tree of *all* occurrences $\underline{\psi}$ of subformulas ψ of φ, where $\underline{\psi}$ is labeled by $v^{\mathrm{G}}_{\mathcal{M}}(\psi)$. In particular, the payoff for myself, and therefore the value of the game for φ coincides with $v^{\mathrm{G}}_{\mathcal{M}}(\varphi)$.

It remains to check that the values labeling formulas of the form $\varphi' \wedge \varphi''$ and $\varphi' \vee \varphi''$ correspond to $\min\{v^{\mathrm{G}}_{\mathcal{M}}(\varphi'), v^{\mathrm{G}}_{\mathcal{M}}(\varphi'')\}$ and $\max\{v^{\mathrm{G}}_{\mathcal{M}}(\varphi'), v^{\mathrm{G}}_{\mathcal{M}}(\varphi'')\}$, respectively. To this aim, we refer to the *polarity* $\pi_\varphi(\underline{\psi}) \in \{+, -\}$ of an occurrence $\underline{\psi}$ of a subformula in φ, defined in a top down manner, as follows:

- $\pi_\varphi(\underline{\varphi}) = +$,
- $\pi_\varphi(\underline{\underline{\psi} \circ \underline{\psi'}}) = \pi_\varphi(\underline{\psi}) = \pi_\varphi(\underline{\psi'})$ for $\circ \in \{\wedge, \vee\}$,
- $\pi_\varphi(\underline{\underline{\psi} \to \underline{\psi'}}) = \pi_\varphi(\underline{\psi'})$; $\pi_\varphi(\underline{\psi}) = -$ if $\pi_\varphi(\underline{\underline{\psi} \to \underline{\psi'}}) = +$ and $\pi_\varphi(\underline{\psi}) = +$ if $\pi_\varphi(\underline{\underline{\psi} \to \underline{\psi'}}) = -$.

It is straightforwardly checked by induction that I am **P** and you are **O** in a state with current formula ψ iff $\pi_\varphi(\underline{\psi}) = +$. For $\psi = \varphi' \vee \varphi''$ this implies that I (as **P**) will choose a subformula labeled by the maximal (truth) value. On the other hand, for $\psi = \varphi' \wedge \varphi''$ you (as **O**) will choose a subformula labeled by the minimal value. The case where $\pi_\varphi(\underline{\psi}) = -$ is dual: I (as **O**) will choose the subformula of $\psi = \varphi' \wedge \varphi''$ that minimizes your (i.e., **P**'s) payoff and therefore maximizes my (**O**'s) own payoff. Likewise, for $\psi = \varphi' \vee \varphi''$ you (as **P**) will choose a subformula with the maximal value. □

REMARK 7.2.2. *If we retain the rule $R^{\mathcal{H}}_{\neg}$ for negation in addition to the (here different) negation defined by $\neg\varphi =_{def} \varphi \to \bot$ we obtain a game for the logic $\mathrm{G}_{\sim}$, which is Gödel logic augmented by involutive negation.*

7.3 An implicit backtracking game for Gödel logic

The $\mathcal{B}$G-game presented above is unsatisfying in a few aspects. As we have already mentioned, no choice of either player is involved in the rule $R^{\mathcal{B}\mathrm{G}}_{\to}$. In fact, if we focus on formulas where implication is the only binary connective, the $\mathcal{B}$G-game can be viewed as just a particular implementation of the evaluation algorithm for G-formulas. Thus a lot of the appeal of game semantics is lost. Another drawback is the comparatively complex way of computing the payoff. In this section we seek to address these worries by defining an alternative semantic game for G where backtracking and thus the use of a stack is left *implicit* in the very same way as a stack for backtracking is implicit in recursive programs: the stack only gets explicit when the recursion is unraveled.

We use $\mathcal{I}\mathrm{G}(\varphi, \omega)$ to denote the *implicit backtracking game* for the logic G ($\mathcal{I}$G-game) for the formula φ, starting with role distribution ω and use $\langle \mathcal{I}\mathrm{G}(\varphi, \omega)\rangle_{\mathbf{P}}$ to denote the value for **P** of that game. (Depending on ω, **P** is either myself or you.) Of course, $\mathcal{I}\mathrm{G}(\varphi, \omega)$ also refers to a given interpretation $\mathcal{M}$. However we prefer to keep that reference implicit in order to simplify notation. Like all other games described in this paper, the $\mathcal{I}$G-game is zero-sum. Modulo this clarification, it is sufficient to mention only the payoff for **P** in the following: the payoff for **O** is always inverse to that for **P**.

The rule for implication in the $\mathcal{I}$G-game is as follows.

($R^{\mathcal{I}\mathrm{G}}_{\to}$) In $\mathcal{I}\mathrm{G}(\varphi \to \psi, \omega)$ **P** chooses whether (1) to continue the game as $\mathcal{I}\mathrm{G}(\varphi, \omega)$ or (2) to play, in addition to $\mathcal{I}\mathrm{G}(\psi, \omega)$, also $\mathcal{I}\mathrm{G}(\varphi, \widehat{\omega})$, where $\widehat{\omega}$ denotes the role distribution that is inverse to ω. In the latter case the payoff for **P** is 1 if $\langle \mathcal{I}\mathrm{G}(\varphi, \omega)\rangle_{\mathbf{P}} \geq \langle \mathcal{I}\mathrm{G}(\psi, \widehat{\omega})\rangle_{\mathbf{P}}$ and -1 otherwise.

REMARK 7.3.1. *While the formulation of $R^{\mathcal{I}\mathrm{G}}_{\to}$ looks quite different from that of the rules for the $\mathcal{B}$Ł- or the $\mathcal{B}$G-game, the difference lies only in the fact that in $R^{\mathcal{I}\mathrm{G}}_{\to}$ we hide details of implementation. If in choice (2) we insist in playing $\mathcal{I}\mathrm{G}(\varphi, \widehat{\omega})$ first and consequently in putting G with $\widehat{\omega}$ on a stack, we obtain a version of the rule that is analogous to those of the earlier games.*

($R^{\mathcal{I}\mathrm{G}}_{\mathrm{at}}$) The payoff for **P** at $\mathcal{I}\mathrm{G}(\varphi, \omega)$ is $v^{\mathrm{G}}_{\mathcal{M}}(\varphi)$.

Note that we do not insist that the game ends upon reaching an atomic formula. Indeed, the payoff may be preliminary since it may only refer to a sub-game of the overall game, as indicated in rule $R^{\mathcal{I}\mathrm{G}}_{\to}$.

The rules for conjunction and disjunction in the $\mathcal{I}$G-game are virtually identical to $R^{\mathcal{H}}_{\wedge}$ and $R^{\mathcal{H}}_{\vee}$ and can be formulated as follows:

($R^{\mathcal{I}\mathrm{G}}_{\wedge}$) In $\mathcal{I}\mathrm{G}(\varphi \wedge \psi, \omega)$ **O** chooses whether to continue the game as $\mathcal{I}\mathrm{G}(\varphi, \omega)$ or as $\mathcal{I}\mathrm{G}(\psi, \omega)$.

($R^{\mathcal{I}\mathrm{G}}_{\vee}$) In $\mathcal{I}\mathrm{G}(\varphi \vee \psi, \omega)$ **P** chooses whether to continue the game as $\mathcal{I}\mathrm{G}(\varphi, \omega)$ or as $\mathcal{I}\mathrm{G}(\varphi, \omega)$.

Remember that no rule for negation is needed because we have $\neg\varphi =_{def} \varphi \to \bot$.

THEOREM 7.3.2. *The value for myself of the* $\mathcal{I}$G*-game for* φ *under the* G*-interpretation* $\mathcal{M}$ *is* w *iff* $v^{\mathrm{G}}_{\mathcal{M}}(\varphi) = w$.

Proof. We show by induction on the complexity of φ that the value $\langle \mathcal{I}\mathrm{G}(\varphi,\omega)\rangle_{\mathbf{P}}$ for **P** of $\mathcal{I}\mathrm{G}(\varphi,\omega)$ is $v^{\mathrm{G}}_{\mathcal{M}}(\varphi)$ for every role distribution ω. (The theorem clearly follows for the role distribution ω where I am **P** and you are **O**.)

According to the rule $R^{\mathcal{I}\mathrm{G}}_{\mathrm{at}}$ the payoff for **P** is $v^{\mathrm{G}}_{\mathcal{M}}(\varphi)$ if φ is atomic. Therefore we have $\langle \mathcal{I}\mathrm{G}(\varphi,\omega)\rangle_{\mathbf{P}} = v^{\mathrm{G}}_{\mathcal{M}}(\varphi)$ in this case.

For the induction step we distinguish the following cases:

$\varphi = \varphi' \wedge \varphi''$: Since **O** can choose whether to continue the game as $\mathcal{I}\mathrm{G}(\varphi',\omega)$ or as $\mathcal{I}\mathrm{G}(\varphi'',\omega)$ and since the payoff for **O** is inverse to that of **P** we obtain $\langle \mathcal{I}\mathrm{G}(\varphi,\omega)\rangle_{\mathbf{P}} = \min\{\langle \mathcal{I}\mathrm{G}(\varphi',\omega)\rangle_{\mathbf{P}}, \langle \mathcal{I}\mathrm{G}(\varphi'',\omega)\rangle_{\mathbf{P}}\}$ and therefore, by the induction hypothesis, $\mathcal{I}\mathrm{G}(\varphi,\omega) = \min\{v^{\mathrm{G}}_{\mathcal{M}}(\varphi'), v^{\mathrm{G}}_{\mathcal{M}}(\varphi'')\}$, as required.

$\varphi = \varphi' \vee \varphi''$: This case is analogous to that for conjunction, except that now the player currently in role **P** can choose how to continue the game. Consequently we obtain $\langle \mathcal{I}\mathrm{G}(\varphi,\omega)\rangle_{\mathbf{P}} = \max\{\langle \mathcal{I}\mathrm{G}(\varphi',\omega)\rangle_{\mathbf{P}}, \langle \mathcal{I}\mathrm{G}(\varphi'',\omega)\rangle_{\mathbf{P}}\}$ and thus $\mathcal{I}\mathrm{G}(\varphi,\omega) = \max\{v^{\mathrm{G}}_{\mathcal{M}}(\varphi'), v^{\mathrm{G}}_{\mathcal{M}}(\varphi'')\}$, as required.

$\varphi = \varphi' \to \varphi''$: If $\langle \mathcal{I}\mathrm{G}(\varphi'',\omega)\rangle_{\mathbf{P}} \geq \langle \mathcal{I}\mathrm{G}(\varphi',\widehat{\omega})\rangle_{\mathbf{P}}$ then by rule $R^{\mathcal{I}\mathrm{G}}_{\to}$ the payoff for **P** and therefore also $\langle \mathcal{I}\mathrm{G}(F,\omega)\rangle_{\mathbf{P}}$ is 1, i.e., optimal for **P**. Consequently **P** will choose to continue the game with the two sub-games $\mathcal{I}\mathrm{G}(\varphi'',\omega)$ and $\mathcal{I}\mathrm{G}(\varphi',\widehat{\omega})$. By the induction hypothesis we have $v^{\mathrm{G}}_{\mathcal{M}}(\varphi') \leq v^{\mathrm{G}}_{\mathcal{M}}(\varphi'')$ in this case, implying $v^{\mathrm{G}}_{\mathcal{M}}(\varphi) = 1$, as required. If, on the other hand, $\langle \mathcal{I}\mathrm{G}(\varphi'',\omega)\rangle_{\mathbf{P}} < \langle \mathcal{I}\mathrm{G}(\varphi',\widehat{\omega})\rangle_{\mathbf{P}}$ then **P** will maximize her payoff by continuing the game as $\mathcal{I}\mathrm{G}(\varphi'',\omega)$. In this case the induction hypothesis implies that $v^{\mathrm{G}}_{\mathcal{M}}(\varphi') > v^{\mathrm{G}}_{\mathcal{M}}(\varphi'')$ and therefore $\langle \mathcal{I}\mathrm{G}(\varphi,\omega)\rangle_{\mathbf{P}} = \langle \mathcal{I}\mathrm{G}(\varphi'',\omega)\rangle_{\mathbf{P}} = v^{\mathrm{G}}_{\mathcal{M}}(\varphi'') = v^{\mathrm{G}}_{\mathcal{M}}(\varphi)$, again as required. □

7.4 An implicit backtracking game for Product logic

Recall that the semantics of Product logic Π is specified by extending a given assignment $v_{\mathcal{M}}$ of values in $[0,1]$ to atomic formulas as follows:

$$v^{\Pi}_{\mathcal{M}}(\varphi \mathbin{\&} \psi) = v^{\Pi}_{\mathcal{M}}(\varphi) \cdot v^{\Pi}_{\mathcal{M}}(\psi)$$

$$v^{\Pi}_{\mathcal{M}}(\varphi \to \psi) = \begin{cases} 1 & \text{if } v^{\Pi}_{\mathcal{M}}(\varphi) \leq v^{\Pi}_{\mathcal{M}}(\psi) \\ v^{\Pi}_{\mathcal{M}}(\psi)/v^{\Pi}_{\mathcal{M}}(\varphi) & \text{otherwise.} \end{cases}$$

Negation is treated as a defined connective via $\neg\varphi =_{def} \varphi \to \bot$, where $v^{\Pi}_{\mathcal{M}}(\bot) = 0$.

One could define a semantic game with explicit backtracking for Π that is very similar to the $\mathcal{B}$Ł-game defined at the beginning of this Section. Roughly speaking one only needs to change the propagation of preliminary payoffs when reaching atomic formulas: instead of addition and subtraction we have to use multiplication and division, respectively. However, as for Gödel logic G above, we prefer to present such a game at a more abstract and compact level that leaves the reference to a game stack and to preliminary payoffs implicit.

The implicit backtracking game for Π ($\mathcal{I}\Pi$-game) for a formula φ starting with role distribution ω is denoted by $\mathcal{I}\Pi(\varphi, \omega)$. By $\langle \mathcal{I}\Pi(\varphi, \omega) \rangle_{\mathbf{P}}$ we denote the value for **P** of that game. Again, we suppress the reference to the underlying interpretation $\mathcal{M}$. Once more, we describe a zero-sum game and thus it is sufficient to specify only the payoff for **P** explicitly.

The implication rule of the $\mathcal{I}\Pi$-game is as follows.

($R^{\mathcal{I}\Pi}_{\rightarrow}$) In $\mathcal{I}\Pi(\varphi \rightarrow \psi, \omega)$ **O** chooses whether (1) to end the game immediately and accept payoff 1 for **P** and -1 for herself or (2) to continue by playing $\mathcal{I}\Pi(\psi, \omega)$ as well as $\mathcal{I}\Pi(F, \widehat{\omega})$, where $\widehat{\omega}$ denotes the role distribution that inverts ω. In this case we have the payoff $\langle \mathcal{I}\Pi(\varphi \rightarrow \psi, \omega)\rangle_{\mathbf{P}} = \langle \mathcal{I}\Pi(\psi, \omega)\rangle_{\mathbf{P}} \,/\, \langle \mathcal{I}\Pi(\varphi, \widehat{\omega})\rangle_{\mathbf{P}}$.

For strong conjunction $\&$, product is used in logic $\Pi$and therefore the following rule will come as no surprise:

($R^{\mathcal{I}\Pi}_{\&}$) In $\mathcal{I}\Pi(\varphi \,\&\, \psi, \omega)$ the game splits into the sub-games $\mathcal{I}\Pi(\varphi, \omega)$ and $\mathcal{I}\Pi(\psi, \omega)$, with total payoff $\langle \mathcal{I}\Pi(\varphi \,\&\, \psi, \omega)\rangle_{\mathbf{P}} = \langle \mathcal{I}\Pi(\varphi, \omega)\rangle_{\mathbf{P}} \cdot \langle \mathcal{I}\Pi(\psi, \omega)\rangle_{\mathbf{P}}$.

Negation is left implicit by $\neg\varphi =_{def} \varphi \rightarrow \bot$.

THEOREM 7.4.1. *The value for myself of the $\mathcal{I}\Pi$-game for φ under the Π-interpretation $\mathcal{M}$ is w iff $v^{\Pi}_{\mathcal{M}}(\varphi) = w$.*

Proof. The proof is very similar to that of Theorem 7.3.2; we show by induction that the value $\langle \mathcal{I}\Pi(F, \omega)\rangle_{\mathbf{P}}$ for **P** of the game $\mathcal{I}\Pi(F, \omega)$ is $v^{\Pi}_{\mathcal{M}}(\varphi)$ for every role distribution ω.

If φ is atomic then the payoff for **P** is $v^{\Pi}_{\mathcal{M}}(\varphi)$ and therefore $\langle \mathcal{I}\Pi(\varphi, \omega)\rangle_{\mathbf{P}} = v^{\Pi}_{\mathcal{M}}(\varphi)$. The induction step for implication and strong conjunction is as follows (the cases for $\varphi = \varphi' \wedge \varphi''$ and for $\varphi = \varphi' \vee \varphi''$ are exactly as in Theorem 7.3.2):

$\varphi = \varphi' \rightarrow \varphi''$: If $\langle \mathcal{I}\Pi(\varphi'', \omega)\rangle_{\mathbf{P}} > \langle \mathcal{I}\Pi(\varphi', \widehat{\omega})\rangle_{\mathbf{P}}$, then $\langle \mathcal{I}\Pi(\varphi'', \omega)\rangle_{\mathbf{P}} \,/\, \langle \mathcal{I}\Pi(\varphi', \widehat{\omega})\rangle_{\mathbf{P}}$ is greater than 1. This implies that in this case **O** achieves a higher payoff by choosing option (1) in rule $R^{\mathcal{I}\Pi}_{\rightarrow}$ and the game ends with the payoff 1 for **P** and thus -1 for **O** herself. On the other hand, if $\langle \mathcal{I}\Pi(\varphi'', \omega)\rangle_{\mathbf{P}} > \langle \mathcal{I}\Pi(\varphi', \widehat{\omega})\rangle_{\mathbf{P}}$, then **O** will choose option (2) and we obtain $\langle \mathcal{I}\Pi(\varphi, \omega)\rangle_{\mathbf{P}} = \langle \mathcal{I}\Pi(\varphi'', \omega)\rangle_{\mathbf{P}} \,/\, \langle \mathcal{I}\Pi(\varphi', \widehat{\omega})\rangle_{\mathbf{P}}$. Finally, if $\langle \mathcal{I}\Pi(\varphi'', \omega)\rangle_{\mathbf{P}} = \langle \mathcal{I}\Pi(\varphi', \widehat{\omega})\rangle_{\mathbf{P}}$ then the choice of **O** is immaterial since the payoff for **P** will always be 1. Clearly, the induction hypothesis yields $\langle \mathcal{I}\Pi(\varphi, \omega)\rangle_{\mathbf{P}} = v^{\Pi}_{\mathcal{M}}(\varphi)$ in all three cases.

$\varphi = \varphi' \,\&\, \varphi''$: By rule $R^{\mathcal{I}\Pi}_{\&}$ we obtain $\langle \mathcal{I}\Pi(\varphi, \omega)\rangle_{\mathbf{P}} = \langle \mathcal{I}\Pi(\varphi', \omega)\rangle_{\mathbf{P}} \cdot \langle \mathcal{I}\Pi(\varphi'', \omega)\rangle_{\mathbf{P}}$ and therefore $\langle \mathcal{I}\Pi(\varphi, \omega)\rangle_{\mathbf{P}} = v^{\Pi}_{\mathcal{M}}(\varphi') \cdot v^{\Pi}_{\mathcal{M}}(\varphi'') = v^{\Pi}_{\mathcal{M}}(\varphi)$ by the induction hypothesis. □

REMARK 7.4.2. *We have only treated propositional logics in this section, but we want to emphasize that all backtracking games presented here can straightforwardly be generalized to the first-order level by adding the rules $R^{\mathcal{H}}_{\forall}$ and $R^{\mathcal{H}}_{\exists}$ defined for the $\mathcal{H}$-game and the $\mathcal{H}$-mv-game in Section 2. As discussed there, this entails making use of the general definition of the value of a game, which refers to optimal payoffs only up to some ϵ.*

8 Propositional random choice games

Following Giles, we have introduced the idea of expected payoffs in a randomized setting in Section 4. However, Giles applied this idea only to the interpretation of *atomic* formulas. For the interpretation of logical connectives and quantifiers in any of the semantic games mentioned so far it does not matter whether the players seek to maximize expected or a certain payoff or, equivalently, try to minimize either expected or certain payments to the opposing player. In [21, 22] it has been shown that considering random choices of witnessing constants in quantifier rules for *Giles-style* games, allows one to model certain (semi-)fuzzy quantifiers that properly extend first-order Łukasiewicz logic. We will take up this idea in Section 9. However in this section we want to explore the consequences of introducing random choices in rules for propositional connectives in the context of *Hintikka-style* games, i.e., games that respect Hintikka's principle, as explained in Section 2.

The results of Section 2 show that, in order to go beyond logic KZ with Hintikka-style games, a new variant of rules has to be introduced. As already indicated, a particularly simple type of new rule, that does not entail any change in the structure of game states, arises from randomization. So far we have only considered rules where either **P** or **O** chooses the sub-formula of the current formula to continue the game with. In game theory one often introduces *Nature* as a special kind of additional player, who does not care what the next state looks like, when it is her time to move and therefore is modeled by a uniformly random choice between all moves available to *Nature* at that state. As we will see below, introducing *Nature* leads to increased expressive power of semantic games. In fact, to keep the presentation of the games simple, we prefer to leave the role of *Nature* only implicit and just speak of random choices, without attributing them officially to a third player. The most basic rule of the indicated type refers to a new propositional connective π and can be formulated as follows.[6]

($R^{\mathcal{R}}_{\pi}$) If the current formula is $\varphi\pi\psi$ then a uniformly random choice determines whether the game continues with φ or with ψ.

REMARK 8.0.1. *Note that no role switch is involved in the above rule: the player acting as **P** remains in this role at the succeeding state; likewise for **O**.*

We call the $\mathcal{H}$-mv-game augmented by rule $R^{\mathcal{R}}_{\pi}$ the *(basic) $\mathcal{R}$-game*. We claim that the new rule gives raise to the following truth function, to be added to the semantics of logic KZ:

$$v_{\mathcal{M}}(\varphi\pi\psi) = (v_{\mathcal{M}}(\varphi) + v_{\mathcal{M}}(\psi))/2.$$

KZ(π) denotes the logic arising from KZ by adding π. To assist a concise formulation of the adequateness claim for the $\mathcal{R}$-game we have to adapt Definition 2.2.1 by replacing 'payoff' with 'expected payoff'. In fact, since we restrict attention to the propositional level here, we can use the following simpler definition.

DEFINITION 8.0.2. *If player **X** has a strategy that leads to an expected payoff for her of at least w, while her opponent has a strategy that ensures that **X**'s expected payoff is at most w, then w is called the* expected value for **X** *of the game.*

THEOREM 8.0.3. *A propositional formula F evaluates to $v_{\mathcal{M}}(\varphi) = w$ in a* KZ(π)*-interpretation $\mathcal{M}$ iff the basic $\mathcal{R}$-game for F with payoffs matching $\mathcal{M}$ has expected value w for myself.*

[6] A similar rule is considered in [54] in the context of partial logic.

Proof. Taking into account that $v_{\mathcal{M}}(\varphi)$ coincides with the value of the $\mathcal{H}$-mv-game matching $\mathcal{M}$ if φ does not contain the new connective π, we only have to add the case for a current formula of the form $\varphi\pi\psi$ to the usual backward induction argument. However, because of the random choice involved in rule $R_{\pi}^{\mathcal{R}}$, it is now her *expected* payoff that **P** seeks to maximize and **O** seeks to minimize.

Suppose the current formula is $\varphi\pi\psi$. By the induction hypothesis, at the successor state σ_φ with current formula φ (the player who is currently) **P** can force[7] an expected payoff $v_{\mathcal{M}}(\varphi)$ for herself, while **O** can force an expected payoff $1 - v_{\mathcal{M}}(\varphi)$ for himself. Therefore the expected value for **P** for the game starting in σ_φ is $v_{\mathcal{M}}(\varphi)$ for **P**. The same holds for ψ instead of φ. Since the choice between the two successor states σ_φ and σ_ψ is uniformly random, we conclude that the expected value for **P** for the game starting with $G\pi H$ is the average of $v_{\mathcal{M}}(\varphi)$ and $v_{\mathcal{M}}(\psi)$, i.e., $(v_{\mathcal{M}}(\varphi) + v_{\mathcal{M}}(\psi))/2$. The theorem thus follows from the fact that I am the initial **P** in the relevant $\mathcal{R}$-game. □

Note that the function $(x + y)/2$ cannot be composed solely from the functions $1 - x$, $\min\{x, y\}$, and $\max\{x, y\}$ and the values 0 and 1. Therefore we can make the following observation.

PROPOSITION 8.0.4. *The connective π is not definable in logic* KZ*.*

But also the following stronger fact holds.

PROPOSITION 8.0.5. *The connective π is not definable in Łukasiewicz logic* Ł*.*

Proof. By McNaughton's Theorem [42] a function $f\colon [0,1]^n \to [0,1]$ corresponds to a formula of propositional Łukasiewicz logic iff f is piecewise linear, where every linear piece has integer coefficients. But clearly the coefficient of $(x + y)/2$ is not an integer. □

REMARK 8.0.6. *We may also observe that, in contrast to* Ł*, not only $\overline{0.5} =_{def} \bot\pi\top$, but in fact every rational number in $[0, 1]$ with a finite (terminating) expansion in the binary number system is definable as a truth constant in logic* KZ(π)*.*

Conversely to Proposition 8.0.5 we also have the following.

PROPOSITION 8.0.7. *None of the connectives $\&$, $\oplus$, $\to$ of* Ł *can be defined in* KZ(π)*.*

Proof. Let Ψ denote the set of all interpretations $\mathcal{M}$, where $0 < v_{\mathcal{M}}(p) < 1$ for all propositional variables p. The following claim can be straightforwardly checked by induction.

Claim: For every formula φ of KZ(π) one of the following holds:

(1) $0 < v_{\mathcal{M}}(\varphi) < 1$ for all $\mathcal{M} \in \Psi$,

(2) $v_{\mathcal{M}}(\varphi) = 1$ for all $\mathcal{M} \in \Psi$, or

(3) $v_{\mathcal{M}}(\varphi) = 0$ for all $\mathcal{M} \in \Psi$.

Clearly this claim does not hold for Ł-formulas of the form $\varphi \,\&\, \psi$, $\varphi \oplus \psi$, and $\varphi \to \psi$. Therefore the connectives $\&$, $\oplus$, $\to$ cannot be defined in KZ(π). □

[7] We re-use the terminology introduced in the proof of Theorem 2.3.4, but applied to *expected* payoffs here.

In light of the above propositions, the question arises whether one can come up with further game rules, that, like $R_\pi^\mathcal{R}$, do not sacrifice what we above called *Hintikka's principle*, i.e., the principle that game state is determined solely by a formula and a role distribution. An obvious way to generalize rule $R_\pi^\mathcal{R}$ is to allow for a (potentially) biased random choice:

($R_{\pi^p}^\mathcal{R}$) If the current formula is $\varphi\pi^p\psi$ then the game continues with φ with probability p, but continues with ψ with probability $1-p$.

Clearly, π coincides with $\pi^{0.5}$. But for other values of p we obtain a new connective. It is straightforward to check that Proposition 8.0.7 also holds if we replace π by π^p for any $p \in [0,1]$.

Interestingly, there is a fairly simple game based way to obtain a logic that properly extends Łukasiewicz logic by introducing a unary connective D that signals that the payoff values for **P** is to be doubled (capped to 1, as usual) at the end of the game.

($R_\mathrm{D}^\mathcal{R}$) If the current formula is $\mathrm{D}\varphi$ then the game continues with φ, but with the following changes at the final state. The payoff, say x, for **P** is changed to $\min\{1, 2x\}$, while the the payoff $1-x$ for **O** is changed to $1-\min\{1,2x\}$.

REMARK 8.0.8. *Instead of explicitly capping the modified payoff for **P** to 1 one may equivalently give **O** the opportunity to either continue that game with doubled payoff for **P** (and inverse payoff for **O** herself) or to simply end the game at that point with payoff 1 for **P** and payoff 0 for **O** herself.*

Let us use $\mathrm{KZ}(\mathrm{D})$ for the logic obtained from KZ by adding the connective D with the following truth function to KZ:

$$v_\mathcal{M}(\mathrm{D}\varphi) = \min\{1, 2 \cdot v_\mathcal{M}(\varphi)\}.$$

Moreover, we use $\mathrm{KZ}(\pi, \mathrm{D})$ to denote the extension of KZ with both π and D and call the $\mathcal{R}$-game augmented by rule $R_\mathrm{D}^\mathcal{R}$ the D-*extended $\mathcal{R}$-game*.

THEOREM 8.0.9. *A propositional formula φ evaluates to $v_\mathcal{M}(\varphi) = w$ in a $\mathrm{KZ}(\pi,\mathrm{D})$-interpretation $\mathcal{M}$ iff the D-extended $\mathcal{R}$-game for φ with payoffs matching $\mathcal{M}$ has expected value w for myself.*

Proof. The proof of Theorem 8.0.3 is readily extended to the present one by considering the additional inductive case of $\mathrm{D}\psi$ as current formula. By the induction hypothesis, the expected value for **P** of the game for G (under the same interpretation $\mathcal{M}$) is $v_\mathcal{M}(\psi)$. Therefore rule $R_\mathrm{D}^\mathcal{R}$ entails that the expected value for **P** of the game for $\mathrm{D}\psi$ is $\min\{1, 2 \cdot v_\mathcal{M}(\psi)\}$. □

Given Proposition 8.0.7 and Theorem 8.0.9 the following simple observation is of some significance.

PROPOSITION 8.0.10. *The connectives &, $\oplus$ and $\rightarrow$ of Ł are definable in $\mathrm{KZ}(\pi, D)$.*

Proof. It is straightforward to check that the following definitions of $\oplus$, &, and $\rightarrow$ as derived connectives in $\mathrm{KZ}(\pi, D)$ match the corresponding truth functions for logic Ł: $\varphi \oplus \psi =_{def} \mathrm{D}(\varphi\pi\psi)$, $\varphi \,\&\, \psi =_{def} \neg\mathrm{D}(\neg\varphi\pi\neg\psi)$, and $\varphi \rightarrow \psi =_{def} \mathrm{D}(\neg\varphi\pi\psi)$. □

REMARK 8.0.11. *Note that Proposition 8.0.10 jointly with Theorem 8.0.9 entails that one can provide game semantics for (an extension of) Łukasiewicz logic* Ł *without dropping Hintikka's principle as in $\mathcal{E}$-games and in $\mathcal{G}$-games.*

REMARK 8.0.12. *The definitions mentioned in the proof of Proposition 8.0.10 give rise to corresponding additional rules for the* D*-extended* $\mathcal{R}$*-game. In particular, for strong disjunction we obtain:*

($R^{\mathcal{R}}_{\oplus}$) *If the current formula is $\varphi \oplus \psi$ then a random choice determines whether to continue the game with φ or with ψ. But in any case the payoff for **P** is doubled (capped to* 1*), while the payoff for **O** remains inverse to that for **P**.*

By further involving role switches similar rules for strong conjunction and for implications are readily obtained. It remains to be seen whether these rules can assist in arguing for the plausibility of the corresponding connective in intended application scenarios. But in any case, it is clear that, compared to the sole specification of truth functions, the game interpretation provides an additional handle for assessing the adequateness of the Łukasiewicz connectives for formalizing reasoning with graded notions and vague propositions.

Like $R^{\mathcal{R}}_{\pi}$, also the rule $R^{\mathcal{R}}_{\mathrm{D}}$ can be generalized in an obvious manner:

($R^{\mathcal{R}}_{\mathrm{M}_c}$) If the current formula is $\mathrm{M}_c\varphi$ then the game continues with φ, but with the following changes at the final state. The payoff, say x, for **P** is changed to $\min\{1, c \cdot x\}$, while the the payoff $1 - x$ for **O** is changed to $1 - \min\{1, c \cdot x\}$.

Adding further instances of π^p and M_c to $\mathrm{KZ}(\pi, \mathrm{D})$ leads to more expressive logics, related to Rational Łukasiewicz Logic and to divisible MV-algebras [24].[8]

REMARK 8.0.13. *Like in Section 7, we have restricted our attention to propositional logics in this section. However, once more, we straightforwardly obtain corresponding first order logics by extending $\mathcal{R}$-game s with Hintikka's original rules $R^{\mathcal{H}}_{\forall}$ and $R^{\mathcal{H}}_{\exists}$ for universal and existential quantification.*

9 Random choice rules for semi-fuzzy quantifiers

As we have seen in the last section, randomization provides a powerful tool for characterizing extensions of logic KZ, i.e., of the 'weak' fragment of Łukasiewicz logic at the propositional level. In this section we will show that allowing for random choices is also useful for characterizing certain types of generalized quantifiers. The simplest quantifier rule of the indicated type can be formulated in direct analogy to the two quantifier rules of the $\mathcal{H}$-game as follows.

($R^{\mathcal{H}}_{\Pi}$) If the current formula is $\Pi x \varphi(x)$ an element c of the domain of $\mathcal{M}$ is chosen randomly and the game continues with $\varphi(c)$.

[8] The following observation (by Petr Hájek) is relevant here: If one adds the truth constant 0.5 to Ł then *all* rational numbers are expressible. Therefore $\mathrm{KZ}(\pi, \mathrm{D})$ extends not only Ł, but also Rational Pavelka Logic, where all rationals truth constants are added to Ł (see [28]). On the other hand, neither, e.g., $\pi^{1/3}$ or M_3 seem to be expressible in $\mathrm{KZ}(\pi, \mathrm{D})$.

Of course, it is not yet clear what exactly we mean by a random choice of a domain element, in particular if the domain is infinite. Fortunately, the intended application of modeling natural language semantics justifies the focus on finite domains: throughout the whole section we will therefore assume that the domain is finite. Moreover we will assume that 'random' means 'uniformly random'. Note that the latter assumption is imperative if we insist that Π is a *logical* quantifier: the meaning of a logical quantifier should be independent of any particular order of domain elements.

While we could add rule $(R^{\mathcal{H}}_{\Pi})$ to the $\mathcal{H}$-game or to the $\mathcal{H}$-mv-game introduced in Section 2, we prefer to switch right away to the more general setting of the $\mathcal{G}$-game (Giles's game) for Łukasiewicz logic Ł as presented in Section 4. In that context rule $R^{\mathcal{H}}_{\Pi}$ has to be reformulated as follows.

$(R^{\mathcal{G}}_{\Pi})$ If the current formula is $\Pi x\varphi(x)$ then the game continues in a state where the indicated occurrence of $\Pi x\varphi(x)$ is replaced by $\varphi(c)$ for some randomly chosen domain element c.

REMARK 9.0.1. *Note that rule $R^{\mathcal{G}}_{\Pi}$ does not refer to the two players (myself and you) at all. Like for the case of the random choice connective π of Section 8, we may think of a third player* Nature *as responsible for the choice. In any case, the rule applies independently of whether $\Pi x\varphi(x)$ is in my or in your tenet.*

It turns out that Π corresponds to a fuzzy quantifier or, more precisely, proportionality quantifier as introduced by Zadeh [55] to model natural language expressions like *few*, *most*, *about a half*, *about ten*, etc. In Zadeh's approach any function from $[0,1]$ into $[0,1]$ may serve as a truth function for a (monadic) fuzzy quantifier that may be applied to an arbitrary formula of some underlying fuzzy logic, in principle. However, as has been repeatedly pointed out in the literature (see, e.g., [22] or the monograph [27]) quite fundamental problems of interpretation arise for quantifiers where not only the quantified formula itself may take an intermediary truth value, but where also the scope of the quantifier occurrence may be fuzzy. Such quantifiers are called type IV quantifiers by Liu and Kerre [38]. To get an idea of the problems alluded to above, consider a formalization of the statement *About half of the visitors are tall* in two different scenarios: in the first scenario half of the visitors are clearly tall and half of them are clearly short, whereas in the second scenario all visitors are of the same height, which is borderline between tall and not tall. Intuitively, we want to accept the statement—i.e, judge it as clearly true—in the first case, but not in the second. However, we cannot distinguish properly between the two indicated scenarios if all that is to be taken into account for computing the truth value of the quantified statement is the 'average' truth value of the instances of the scope formula. For this reason we avoid type IV quantifiers and focus on so-called semi-fuzzy quantifiers (type III quantifiers in the classification of [38]), where the scope is always a crisp, i.e., classical, formula and not just any Ł-formula. Formally, we specify the language for an extension Ł(Qs), of first order Łukasiewicz logic Ł, where Qs is a list of (unary) quantifier symbols other than $\forall$ or $\exists$, as follows:

$$\gamma \ ::= \ \bot \mid \hat{P}(\vec{t}) \mid \neg\gamma \mid (\gamma \vee \gamma) \mid (\gamma \wedge \gamma) \mid \forall v\gamma \mid \exists v\gamma$$

$$\varphi \ ::= \ \gamma \mid \tilde{P}(\vec{t}) \mid \neg\varphi \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \to \varphi) \mid (\varphi \,\&\, \varphi) \mid \forall v\varphi \mid \exists v\varphi \mid \mathsf{Q}v\gamma,$$

where $\hat{P}$ and $\tilde{P}$ are meta-variables for classical and for general (i.e., possibly fuzzy) predicate symbols, respectively, $\mathsf{Q} \in Qs$; v is our meta-variable for object variables; $\vec{t}$ denotes a sequence of terms, i.e., either object variable or constant symbol, matching the arity of the preceding predicate symbol. Note the scope of the additional quantifiers from Qs is always a classical formula. Otherwise the syntax is as for Ł itself.

The following notion supports a crisp specification of truth functions for semi-fuzzy proportionality quantifiers over finite interpretations.

DEFINITION 9.0.2. *Let $\psi(x)$ be a classical formula and $v_{\mathcal{M}}(\cdot)$ a corresponding evaluation function over the finite domain D. Then*

$$\mathrm{Prop}_x\, \psi(x) = \frac{\sum_{c \in D} v_{\mathcal{M}}(\psi(c))}{|D|}.$$

$\mathrm{Prop}_x\, \psi(x)$ thus denotes the proportion of all elements in D satisfying the classical formula ψ. Remember that we stipulated above that all random choices are made with respect to a uniform probability distribution over D. Therefore $\mathrm{Prop}_x\, \psi(x)$ denotes the probability that a randomly chosen element satisfies ψ.

The following theorem generalizes Theorem 4.0.1 and states that rule $R_{\Pi}^{\mathcal{G}}$ matches the extension of the valuation function for Ł to Ł(Π) by

$$v_{\mathcal{M}}(\Pi x \psi(x)) = \mathrm{Prop}_x\, \psi(x).$$

THEOREM 9.0.3. *A Ł(Π)-sentence φ is evaluated to $v_{\mathcal{M}}(\varphi) = w$ in an interpretation $\mathcal{M}$ iff the $\mathcal{G}$-game for φ augmented by rule $R_{\Pi}^{\mathcal{G}}$ has value $1 - w$ for myself under risk value assignment $\langle \cdot \rangle_{\mathcal{M}}$.*

Theorem 9.0.3 is an instance of Theorem 9.2.2, proved below.

9.1 A note on binary quantifiers

Natural language quantifiers are usually binary, as in *About half of the students are present*, rather than unary as in *About half* [*of the elements in the domain of discourse*] *are present*. However, binary quantifiers like *about half*, *many*, *at least a third*, etc, are extensional. This means that, like in the above example, the first argument of the binary quantifier—its range—is only used to restrict the universe of discourse. More formally, let $\hat{\psi}$ denote the set of domain elements that satisfy the (crisp) predicate expressed by the classical formula $\psi(x)$. If Q is a unary quantifier, then ${}^{\psi}\mathsf{Q}x\varphi(x)$ is a quantified statement defined by $v_{\mathcal{M}}({}^{\psi}\mathsf{Q}x\varphi(x)) = v_{\mathcal{M}'}(\mathsf{Q}x\varphi(x))$, where $\mathcal{M}'$ denotes the interpretation that results from $\mathcal{M}$ by restricting the domain of $\mathcal{M}$ to $\hat{\psi}$. This reduces extensional binary quantification to unary quantification, here illustrated for Π as follows:

($R_{\Pi^2}^{\mathcal{G}}$) Asserting ${}^{\psi}\Pi x\varphi(x)$ reduces to asserting $\varphi(c)$ where c is a (uniformly) randomly chosen element of $\hat{\psi}$: $\varphi(c)$ replaces ${}^{\psi}\Pi x\varphi(x)$ in the corresponding tenet.

If the classical formula $\psi(x)$ is atomic then it is clear what it means to randomly choose an element of $\hat{\psi}$ (unless $\hat{\psi}$ is empty). However, if $\psi(x)$ is of arbitrary logical complexity, then we may remain within our game semantic framework by employing the $\mathcal{H}$-game of Section 2 to find an appropriate random witness element as follows:

1. Choose a random domain element c.
2. Initiate an $\mathcal{H}$-game where a Proponent **P** defends $\psi(c)$ against an Opponent **O**.
3. If **P** wins the $\mathcal{H}$-game, then the main $\mathcal{G}$-game is continued with the constant c, i.e., with $\varphi(c)$ replacing ${}^{\psi}\Pi x\varphi(x)$. Otherwise, return to 1.

Note that it is important to keep the objectives of the players **P** and **O** in the $\mathcal{H}$-game independent from the objectives of the players in the $\mathcal{G}$-game. By Theorem 2.1.1 **P** wins the $\mathcal{H}$-game against the rational Opponent **O** if and only if the classical formula $\psi(c)$ is true, i.e., if $c \in \psi$. Note that the indicated procedure and therefore the main $\mathcal{G}$-game will fail to terminate if the range ψ is empty. This is in accordance with the above definition that leaves $v_{\mathcal{M}}({}^{\psi}\mathsf{Q}x\varphi(x))$ undefined if the range is empty. According to the classic linguistic paper [5] by Barwise and Cooper this matches intuitions about natural language quantifiers applied to an empty range.

9.2 Blind choice quantifiers

Remember that in the context of the $\mathcal{G}$-game we have considered three types of challenges to the defender **X** of a quantified sentence $\mathsf{Q}x\psi(x)$. In each case **X** has to assert $\psi(c)$, but the constant (domain element) is either

(A) chosen by the attacker (i.e., by the current opponent **O**),

(D) chosen by the defender (i.e., by the current proponent **P**), or

(R) chosen randomly (i.e., by *Nature*).

We will speak of a challenge of type A, D, or R, respectively. With respect to a formula $\mathsf{Q}x\psi(x)$ we will say that the two players (myself and you) either bet *for* or *against* $\psi(c)$. Betting for $\psi(c)$ simply means to assert $\psi(c)$, betting against $\psi(c)$ is equivalent to betting for $\neg\psi(c)$ and thus amounts to an assertion of ψ in exchange for an assertion of $\psi(c)$ by the opposing player. We interpret the latter bet as follows: **X** pays 1€ for a ticket that entitles her to receive whatever payment by her opponent **Y** is due for **Y**'s assertion of $\psi(c)$ according to the results of associated dispersive experiments made at the end of the game.

Like in Section 5 for proportional rules in Giles's game, we will speak of a *round* of a game as consisting of a player's *attack* of an assertion made by the other player, followed by a *defense* of that latter player. Moreover, we recall from Section 5 that by the principle of limited liability for defense (LLD) asserting $\perp$ is always a valid defense move. Moreover, by the other form of the principle of limited liability (LLA), the opponent, instead of attacking an assertion in some specific way, may grant the assertion which will consequently be deleted from the current state of the game. In general, when an assertion of $\mathsf{Q}x\psi(x)$ is attacked, the round results in a state where both players are placing certain numbers of bets for or against various instances of $\psi(x)$, where the challenge determining the constants replacing x can be of type A, D, or R. In this manner we arrive at a rich set of possible quantifier rules. However, here we are only interested in type R challenges. We will call $\psi(c)$ a random instance of $\psi(x)$ if c has been chosen randomly.

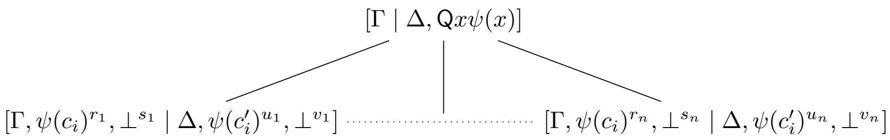


Figure 6. Schematic blind choice quantifier rule — my possible defenses to a particular attack by you.

We first investigate the family of *blind choice quantifiers*, defined as follows.

DEFINITION 9.2.1. Q *is a* (semi-fuzzy) blind choice quantifier *if it can be specified by a game rule satisfying the following two conditions:*

(i) Only challenges of type R are allowed: an attack on $\mathsf{Q}x\psi(x)$ *followed by a defense move results in a state where both players have placed a certain number (possibly zero) of bets* for *and* against *random instances of* $\psi(x)$.

(ii) The identity of the random constants is revealed to the players only at the end of the round, i.e., after an attack has been chosen by the one player and a corresponding defense move has been chosen by the other player.

Like in all other game rules the occurrence of the attacked formula is removed from the corresponding tenet.

Figure 6 depicts possible state transitions involved in the application of a blind choice quantifier rule, where I am the proponent (defender) of the formula $\mathsf{Q}x\psi(x)$. Γ and Δ denote arbitrary multisets of formulas; the classical formula $\psi(x)$ forms the scope of $\mathsf{Q}x\psi(x)$ asserted by myself and attacked by you; $\bot^k$ denotes k occurrences of $\bot$; and $\psi(c_i)^k$ is used as an abbreviation for the k assertions of random instances $\psi(c_1), \ldots, \psi(c_k)$. Note that in general there is more than one way in which you may attack my assertion of $\mathsf{Q}x\psi(x)$. Figure 6 only shows the scheme for one particular attack. A presentation of a full rule consists of a finite number of instances of this scheme. The root of all these trees is labeled by $[\Gamma \mid \Delta, \mathsf{Q}x\psi(x)]$, which means that the effect of the different attacks is shown only at the end of a full round, i.e., only after also a corresponding defense has been chosen. The principle of limited liability LLA implies that you, the attacker, may choose to simply remove the exhibited occurrence of $\mathsf{Q}x\psi(x)$ from the state. In other words, every rule includes an instance of Figure 6 that consists of only one branch ($n = 1$), where $r_1 = s_1 = u_1 = v_1 = 0$. For myself as defender, the principle of limited liability LLD implies that in any other instance of the schematic tree there is a branch i with $r_i = s_i = u_i = 0$ and $v_i = 1$, i.e., where I reply to your attack by asserting $\bot$.

We assume that for every rule for my assertion of a formula, there is a corresponding rule for your assertion of the same formula, that arises by switching our roles. It therefore suffices to investigate explicitly only those rules, where I am in role **P**, i.e., for occurrences of quantified formulas in my tenet.[9]

[9] Formulating a rule only for myself in the role of **P** has the advantage that we do not have to consider two cases when we specify the game states that result from applying the rule.

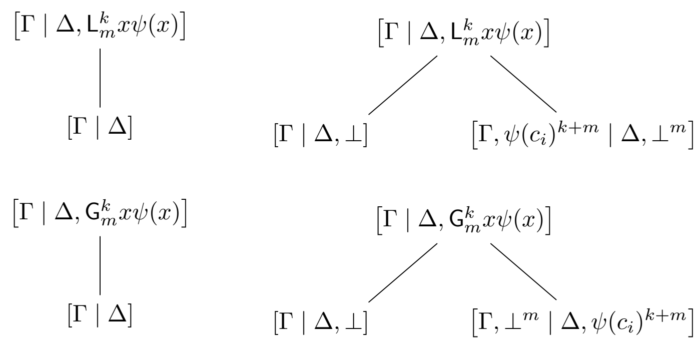


Figure 7. Games rules $R_{\mathsf{L}^k_m}$ and $R_{\mathsf{G}^k_m}$

We specify two concrete families of blind choice quantifiers, L^k_m and G^k_m, for every $k, m \geq 1$, by the following rules:

($R_{\mathsf{L}^k_m}$) If I assert $\mathsf{L}^k_m x\psi(x)$ then you may attack by betting for k random instances of $\psi(x)$, while I bet against m random instances of $\psi(x)$.

($R_{\mathsf{G}^k_m}$) If I assert $\mathsf{G}^k_m x\psi(x)$ then you may attack by betting against m random instances of $\psi(x)$, while I bet for k random instances of $\psi(x)$.

We insist on condition *(ii)* of Definition 9.2.1: the random constants used to obtain the mentioned instances of $\psi(x)$ are only revealed to the players after they have placed their bets. Moreover, although not explicitly mentioned, the principle of limited liability remains in force in both forms (LLA and LLD). Therefore, by LLA, I as the defender (i.e., in role **P**) may also respond to your attack by asserting $\bot$. On the other hand, by LLD, you may grant the formula occurrence in question without attacking it. It is then simply removed from my tenet. However, if none of the players invokes the principle of limited liability the following successor game states are reached:

$$\text{for } \mathsf{L}^k_m x\psi(x): \quad [\Gamma, \psi(c_i)^{k+m} \mid \Delta, \bot^m]$$
$$\text{for } \mathsf{G}^k_m x\psi(x): \quad [\Gamma, \bot^m \mid \Delta, \psi(c_i)^{k+m}] .$$

Consequently, the rules for my assertion of $\mathsf{L}^k_m x\psi(x)$ or of $\mathsf{G}^k_m x\psi(x)$ can be depicted as shown in Figure 7. The rules for your assertion of $\mathsf{L}^k_m x\psi(x)$ or of $\mathsf{G}^k_m x\psi(x)$ are analogous.

We claim that these rules match the extension of Ł to Ł$(\mathsf{L}^k_m, \mathsf{G}^k_m)$ by

$$v_{\mathcal{M}}(\mathsf{L}^k_m x\psi(x)) = \min\{1, \max\{0, 1 + k - (m+k)\,\mathrm{Prop}_x\,\psi(x)\}\} \tag{8}$$
$$v_{\mathcal{M}}(\mathsf{G}^k_m x\psi(x)) = \min\{1, \max\{0, 1 - k + (m+k)\,\mathrm{Prop}_x\,\psi(x)\}\}. \tag{9}$$

THEOREM 9.2.2. *A* Ł$(\mathsf{L}^k_m, \mathsf{G}^k_m)$*-sentence φ is evaluated to $v_{\mathcal{M}}(\varphi) = x$ in an interpretation $\mathcal{M}$ iff every $\mathcal{G}$-game for φ augmented by the rules $R_{\mathsf{L}^k_m}$ and $R_{\mathsf{G}^k_m}$ is has value $1 - x$ for me under risk value assignment $\langle\cdot\rangle_{\mathcal{M}}$.*

Proof. Relative to the proof of Theorem 4.0.1 (see [19, 25, 26]) we only have to consider states of the form $\left[\Gamma \mid \Delta, \mathsf{L}^k_m x\psi(x)\right]$ and $\left[\Gamma \mid \Delta, \mathsf{G}^k_m x\psi(x)\right]$. (I.e., we only consider situations where my assertion of an L^k_m- or G^k_m-quantified sentences is considered for attack by you. The cases for your assertions of $\mathsf{L}^k_m x\psi(x)$ or $\mathsf{G}^k_m x\psi(x)$ are dual.) In fact, since G^k_m is treated analogously to L^k_m, we may focus on states of the form $\left[\Gamma \mid \Delta, \mathsf{L}^k_m x\psi(x)\right]$ without loss of generality. Like for the other connectives, we obtain the total risk at such a state as the sum of the risk for the exhibited assertion and of the risk for the rest of the state:

$$\left\langle \Gamma \mid \Delta, \mathsf{L}^k_m x\psi(x)\right\rangle = \langle \Gamma \mid \Delta \rangle + \left\langle \ \mid \mathsf{L}^k_m x\psi(x)\right\rangle .$$

It remains to show that the reduction of the exhibited quantified formula to instances according to rule $(R_{\mathsf{L}^k_m})$ results in a risk that corresponds to the specified truth function if we play rationally. According to Figure 7 the three possible successor states are $\left[\psi(c_i)^{k+m} \mid \bot^m\right]$, $[\ \mid\ ]$, and $[\ \mid \bot]$, respectively. In the first case, revealing the constants to the players also reveals the amount of money I have to pay, since only classical formulas are involved: I have to pay m€ to you for my m assertions of $\bot$, while for each of your $k+m$ assertions you have to pay me either 0€ or 1€. In total I have to pay to you between $-k$€ and m€, depending on the random constants c_i. The risk value of the game state *before* the identities of the constants are revealed to the players is therefore calculated as the *expected* value for this amount. It is binomially distributed and readily computed as

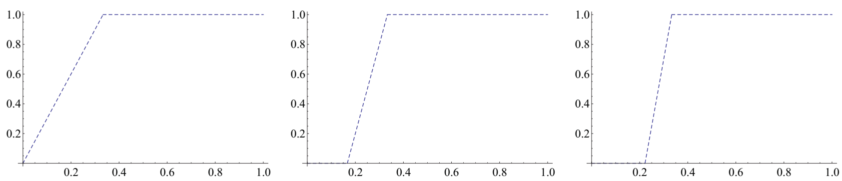
$$m - \sum_{i=0}^{k+m} i \cdot (\mathrm{Prop}_x\, \psi(x))^{k+m-i}(1-\mathrm{Prop}_x\, \psi(x))^i \binom{k+m}{i} =$$
$$m-(k+m)(1-\mathrm{Prop}_x\,\psi(x)) = -k+(k+m)\,\mathrm{Prop}_x\,\psi(x)).$$

The second case (state $[\ \mid\ ]$, carrying risk 0) arises if you choose to grant my assertion of the formula, which for you is the rational choice if the above expression is below 0. The third case (state $[\ \mid \bot]$, carrying risk 1) arises if I invoke the principle of limited liability LLD to hedge my expected loss. Thus we obtain

$$\left\langle \ \mid \mathsf{L}^k_m x\psi(x)\right\rangle = \min\{1, \max\{0, -k+(k+m)\,\mathrm{Prop}_x\,\psi(x)\}\} = 1 - v_{\mathcal{M}}(\mathsf{L}^k_m x\psi(x)),$$

which means that the claimed correspondence between the truth function and the risk resulting from playing rationally holds. □

At least about a third. For illustration, let us take a closer look at quantifiers of the form G^s_{2s}. We argue that these quantifiers can be used to model the natural language expression *at least about a third*. Note that the attacker of $\mathsf{G}^s_{2s} x\psi(x)$ is supposed to believe that $\psi(x)$ holds for clearly less than a third of all domain elements (otherwise she would grant the assertion). Consequently she is willing to place $2s$ bets *against* random instances of $\psi(x)$ if the defender places s bets *for* such random instances. Figure 8 shows the resulting truth functions for sample sizes $(2s+s)$ $3, 6$, and 9, where the horizontal axis corresponds to $\mathrm{Prop}_x\,\psi(x)$ and the vertical axis to $v_{\mathcal{M}}(\mathsf{G}^s_{2s} x\psi(x))$. Functions like these are routinely suggested to represent natural language quantifiers like *at least about*



(a) $\mathsf{G}^1_2 x\psi(x)$ (b) $\mathsf{G}^2_4 x\psi(x)$ (c) $\mathsf{G}^3_6 x\psi(x)$

Figure 8. Truth functions for $\mathsf{G}^s_{2s} x\psi(x)$

a third in the fuzzy logic literature.[10] However no justification beyond intuitive plausibility is usually given. In contrast, our model allows one to extract such truth functions from an underlying semantic principle: namely the willingness to bet that sufficiently many randomly chosen witnesses support or put into doubt the relevant statement.

As noted above, the quantifiers L^k_m and G^k_m are only (very restricted) examples of blind choice quantifiers. Nevertheless, they turn out to be expressive enough to define *all* blind choice quantifiers in the context of Kleene–Zadeh logic KZ (weak Łukasiewicz logic):

THEOREM 9.2.3. *All blind choice quantifiers can be expressed using quantifiers of the form L^k_m and G^k_m, conjunction $\wedge$, disjunction $\vee$, and $\bot$.*

Proof. As illustrated in Figure 6 above, the game state resulting from an attack and a corresponding defense of my assertion of a blind choice quantifiers is always of the form $[\Gamma, \psi(c_i)^r, \bot^s \mid \Delta, \psi(c_i')^u, \bot^v]$. Analogously to the proof of Theorem 9.2.2, the associated risk before the identities of the constants are revealed is computed as

$$\langle \Gamma \mid \Delta \rangle + v - s + (u - r)(1 - \mathrm{Prop}_x\, \psi(x)).$$

Remember that $\psi(c_i)^k$ is short hand notation for k (in general) different random instances of $\psi(x)$. As a first step towards a simplified uniform presentation of arbitrary blind choice quantifiers, note the following. Instead of picking $u + r$ random constants we can rather investigate the game state $[\Gamma, \psi(c)^r, \bot^s \mid \Delta, \psi(c)^u, \bot^v]$ where only one random constant c is picked, since this modification does not change the *expected* risk. As a further step, note that game states where assertions of $\psi(c)$ are made by *both* players show redundancies in the sense that there are equivalent game states where $\psi(c)$ occurs only in one of the two multisets of assertions that represent a state. Likewise for game states with assertions of $\bot$ made by both players. Depending on v, s, u, and r, an equivalent game state is given by:

$$\begin{array}{ll}
(1) & \left[\Gamma, \psi(c)^{r-u} \mid \Delta, \bot^{v-s}\right] \text{ if } v > s \text{ and } r > u \\
(2) & \left[\Gamma, \psi(c)^{r-u}, \bot^{s-v} \mid \Delta\right] \text{ if } v \leq s \text{ and } r > u \\
(3) & \left[\Gamma \mid \Delta, \psi(c)^{u-r}, \bot^{v-s}\right] \text{ if } v > s \text{ and } r \leq u \\
(4) & \left[\Gamma, \bot^{s-v} \mid \Delta, \psi(c)^{u-r}\right] \text{ if } v \leq s \text{ and } r \leq u.
\end{array}$$

[10] For example in [27] trapezoidal functions like the ones in Figure 8 are explicitly suggested for natural language quantifiers of this kind.

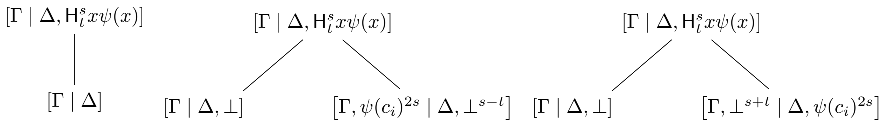
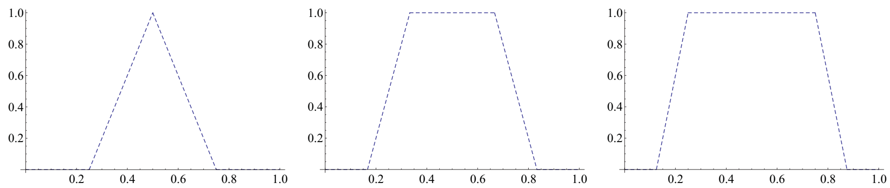


Figure 9. The rule $R_{\mathsf{H}^s_t}$



(a) $\mathsf{H}^2_0 x\psi(x)$ (b) $\mathsf{H}^3_1 x\psi(x)$ (c) $\mathsf{H}^4_2 x\psi(x)$

Figure 10. Truth functions for $\mathsf{H}^s_t x\psi(x)$

Note that states of type (2) are redundant: Playing rationally, you will invoke the principle of limited liability LLA and grant the quantified formula, rather than make an assertion without being compensated by any assertions made by myself. The resulting state is $[\Gamma \mid \Delta]$ in this case. On the other hand, states of type (3) reduce to state $[\Gamma \mid \Delta, \bot]$, since I may invoke the principle of liability LLD. For states of type (1) I will invoke principle LLD if $v - s > r - u$. Similarly, you will invoke principle LLA to ensure that only those states of type (4) have to be considered where $s - v \leq u - r$. For appropriate choices of k and m, this leaves us with states that result from the rules for either $\mathsf{L}^k_m x\psi(x)$ or for $\mathsf{G}^k_m x\psi(x)$.

Finally observe that all of my defenses to your attack on $\mathsf{Q}x\psi(x)$ lead to successor states which are reached also by suitable instances of $\mathsf{G}^k_m x\psi(x)$, of $\mathsf{L}^k_m x\psi(x)$ or of $\bot$. Hence my risk for that attack amounts to the minimum of the risk values for these successor states, which in turn equals the risk value for asserting the disjunction of these instances. Similarly, since you can choose between several attacks on $\mathsf{Q}x\psi(x)$ in the first place, my risk for $\mathsf{Q}x\psi(x)$ amounts to the maximum of the risks for these attacks. Hence it is equal to the risk of the conjunction of these disjunctions. □

About half. As an example consider the family of quantifiers H^s_t, defined by the game rule depicted in Figure 9.

We suggest that H^s_t induces plausible fuzzy models for the natural language quantifier *about half*. Figure 10 shows the truth functions for three different quantifiers of this family, where the horizontal axis corresponds to $\mathrm{Prop}_x\, \psi(x)$ and the vertical axis to $v_{\mathcal{M}}(\mathsf{H}^s_t x\psi(x))$.

The two parameters of H^s_t can be interpreted as follows: s determines the sample size (i.e., the number of random instances involved in reducing the quantified formula), while t may be called the *tolerance*, since the smaller t gets, the closer $\mathrm{Prop}_x\, \psi(x)$ has to be to $1/2$ if $\mathsf{H}^s_t x\psi(x)$ is to be evaluated as perfectly true. If $t = 0$ (zero tolerance) then $v_{\mathcal{M}}(\mathsf{H}^s_0 x\psi(x)) = 1$ if only if $\mathrm{Prop}_x\, \psi(x) = 1/2$ in M. By increasing t (while maintaining the same sample size s) the range of values for $\mathrm{Prop}_x\, \psi(x)$ that guarantee $v_{\mathcal{M}}(\mathsf{H}^s_0 x\psi(x)) = 1$ grows symmetrically around $1/2$.

As an instance of Theorem 9.2.3 we obtain that $\mathsf{H}^s_t x\psi(x)$ is equivalent to the formula $\mathsf{G}^{s-t}_{s+t} x\psi(x) \wedge \mathsf{L}^{s+t}_{s-t} x\psi(x)$. The tree at the center of Figure 9 corresponds to the rule for G^{s-t}_{s+t} and the one at the right hand side corresponds to the rule for L^{s+t}_{s-t}. The tree at the left hand side corresponds to the fact that the attacker may choose to grant the formula.

Next we show how arbitrary blind choice quantifiers can be reduced to the quantifier Π introduced at the beginning of this section if implication, negation, and strong disjunction, but also truth constants that evaluate to certain rational numbers are available in the language. (This actually corrects an error in [22]).

THEOREM 9.2.4. *The blind choice quantifiers L^k_m and G^k_m for all $m, k \geq 1$ can be expressed in* Ł(Π) *enriched by certain truth constants via the following reductions:*

$$v_{\mathcal{M}}(\mathsf{L}^k_m x\psi(x)) = v_{\mathcal{M}}([\neg(\overline{(1+k)/(m+k)} \to \Pi x\psi(x))]^{m+k}_{\oplus})$$
$$v_{\mathcal{M}}(\mathsf{G}^k_m x\psi(x)) = v_{\mathcal{M}}([\neg(\Pi x\psi(x) \to \overline{(k-1)/(m+k)})]^{m+k}_{\oplus})$$

where $[\phi]^n_{\oplus}$ denotes $\phi \oplus \ldots \oplus \phi$, n times, and $\overline{a}$ denotes the truth constant for $a \in [0,1]$.

Proof. Note that the truth functions of $\mathsf{G}^k_m x\psi(x)$ and $\mathsf{L}^k_m x\psi(x)$ depend only on the value of $\mathrm{Prop}_x\, \psi(x)$, while the random choice quantifier Π is directly represented by the truth function $\mathrm{Prop}_x\, \psi(x)$. Hence the equivalences can easily be checked by computing the truth value of the respective right hand side formula and comparing it to the truth function for the corresponding quantifier. □

COROLLARY 9.2.5. *All blind choice quantifiers can be expressed in* Ł(Π) *enriched by rational truth constants.*

The corollary follows directly from Theorems 9.2.3 and 9.2.4.

9.3 Deliberate choice quantifiers

In the previous section we surveyed the family of blind choice quantifiers and concluded that these quantifiers all amount to piecewise linear truth functions. A much more general class of quantifiers arises by dropping condition *(ii)* of Definition 9.2.1. As an example of this class we investigate the family of so-called *deliberate choice quantifiers*, specified by the following schematic game rule, where ψ is a classical formula:

($R_{\Pi^k_m}$) If I assert $\Pi^k_m x\psi(x)$ and you decide to attack, then $k+m$ (not necessarily different) constants are chosen randomly and I have to pick k of those constants, say $c_1, \ldots, c_k$, and bet for $\psi(c_1), \ldots, \psi(c_k)$, while simultaneously betting against $\psi(c'_1), \ldots, \psi(c'_m)$, where $c'_1, \ldots, c'_m$ are the remaining m random constants.

(Recall that the scheme for your assertions of $\Pi^k_m x\psi(x)$ arises from switching our roles.) Although not mentioned explicitly, we emphasize that principle of limited liability remains in place: after the constants are chosen, by LLD, I may assert $\bot$ (i.e., agree to pay 1€) instead of betting as indicated above. Therefore I have $1 + \binom{k+m}{k}$ possible defenses to your attack on my assertion of $\Pi^k_m x\psi(x)$: either I choose to hedge my loss by asserting $\bot$ or I pick k out of the $k+m$ random constants to proceed as indicated.

We claim that this rule matches the extension of Ł to Ł(Π^k_m) by

$$v_{\mathcal{M}}(\Pi^k_m \psi(x)) = \binom{k+m}{k} (\mathrm{Prop}_x\, \psi(x))^k (1 - \mathrm{Prop}_x\, \psi(x))^m.$$

THEOREM 9.3.1. *A Ł(Π^k_m)-sentence φ is evaluated to $v_{\mathcal{M}}(\varphi) = x$ in interpretation $\mathcal{M}$ iff every $\mathcal{G}$-game for φ augmented by rule $R_{\Pi^k_m}$ has value $1 - x$ for myself under risk value assignment $\langle \cdot \rangle_{\mathcal{M}}$.*

Proof. Like in the proof of Theorem 9.2.2, we only have to consider states of the form $\left[\Gamma \mid \Delta, \Pi^k_m x\psi(x)\right]$. Again, we can separate the risk for the exhibited assertion from the risk for the remaining assertions:

$$\left\langle \Gamma \mid \Delta, \Pi^k_m x\psi(x) \right\rangle = \langle \Gamma \mid \Delta \rangle + \left\langle \;\mid \Pi^k_m x\psi(x) \right\rangle.$$

It remains to show that my optimal way to reduce the exhibited quantified formula to instances as required by rule $(R_{\Pi^k_m})$ results in a risk that corresponds to the specified truth function. For the following argument remember that the principle of limited liability is in place. Moreover remember that $\psi(x)$ is classical. This means that I either finally have to pay 1€ for my assertion of $\Pi^k_m x\psi(x)$ or do not have to pay anything at all for it. The latter is only the case if all my bets for $\psi(c_1), \ldots, \psi(c_k)$, as well as all my bets against $\psi(c'_1), \ldots, \psi(c'_m)$, for $c_1, \ldots, c_k$, $c'_1, \ldots, c'_m$ as specified in rule $(R_{\Pi^k_m})$, succeed. Let the random variable K denote the number of chosen elements c on which my bet is successful, i.e., where $\langle \psi(c) \rangle = 0$. Then K is binomially distributed and the probability that this event obtains (the inverse of my associated risk) is readily calculated to be

$$\binom{k+m}{k} \mathrm{Prop}_x\, \psi(x)^k (1 - \mathrm{Prop}_x\, \psi(x))^m. \qquad \square$$

At a first glance, the deliberate choice quantifier Π^k_m might seem suitable for modeling the natural language quantifier *about* k *out of* $m+k$. However, a look at the corresponding graph for $\left\langle \Pi^1_1 x\psi(x) \right\rangle$ (Figure 11) reveals that the risk for asserting $\Pi^1_1 x\psi(x)$ is always larger than 0.5. Therefore the statement is never more than just 'half-true'. This is clearly not in accordance with intuitions about the truth conditions for statements like *About half of the doors are locked* if, say, 49 out of 100 are locked.

An additional mechanism is needed to obtain more appropriate models of natural quantifier expressions like *about half*. While there are many ways to achieve the desired effect, we confine ourselves here to a particularly simple operator that nicely fits our semantic framework, since it arises by simply multiplying involved bets. Given a number $n \geq 2$ and a semi-fuzzy quantifier Q we specify the semi-fuzzy quantifier $\mathsf{W}_n(\mathsf{Q})$ by the following rule.

$(\mathsf{W}_n(\mathsf{Q})x\psi(x))$ If I assert $\mathsf{W}_n(\mathsf{Q})x\psi(x)$ then you have to place n bets *against* $\mathsf{Q}x\psi(x)$ while I have to bet *for* $\mathsf{Q}x\psi(x)$ just once. (Analogously for your assertion of $\mathsf{W}_n(\mathsf{Q})x\psi(x)$.)

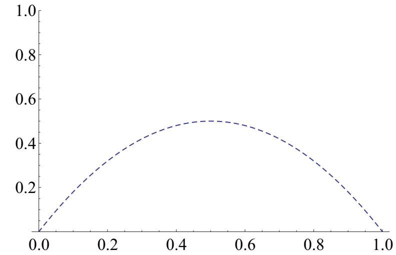
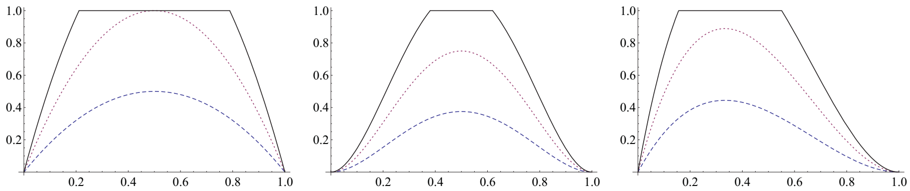


Figure 11. Truth value for $\Pi^1_1 x\psi(x)$ (depending on p)



(a) $\mathsf{W}_i(\Pi^1_1)x\psi(x)$ (b) $\mathsf{W}_i(\Pi^2_2)x\psi(x)$ (c) $\mathsf{W}_i(\Pi^1_2)x\psi(x)$

Figure 12. W_i-modified proportionality quantifiers; the graphs correspond to the cases $i = 1$, $i = 2$, and $i = 3$ from bottom to top in each diagram.

Note that W_n is acting here as a *quantifier modifier*; for any semi-fuzzy quantifier Q, $\mathsf{W}_n(\mathsf{Q})$ still denotes a semi-fuzzy quantifier. Furthermore the principle of limited liability remains in place, hence the game state $\langle \Gamma \mid \Delta, \mathsf{W}_n(\mathsf{Q})x\psi(x)\rangle$ is reduced to $\left\langle \Gamma, \bot^n \mid \Delta, \mathsf{Q}x\psi(x)^{n+1}\right\rangle$ or to $\langle \Gamma \mid \Delta\rangle$, depending on whether it is is preferable from the attacker's point of view to attack or to grant the assertion of $\mathsf{Q}x\psi(x)^{n+1}$. (The defender never has to invoke the principal of limited liability in optimal strategies.) Moreover, similarly as in Theorem 9.2.4, W_n can be expressed using negation and strong conjunction by

$$v_{\mathcal{M}}(\mathsf{W}_n(\mathsf{Q})x\psi(x)) = v_{\mathcal{M}}(\neg(\neg\mathsf{Q}x\psi(x))^{n+1}).$$

For the the truth functions for some of the quantifiers of type $\mathsf{W}_n(\Pi^k_m)$ see Figure 12.

The quantifier $\mathsf{W}_3(\Pi^2_2)$ may be considered as formal fuzzy counterpart of the informal expression *about half*. Likewise, $\mathsf{W}_3(\Pi^1_2)$ may be understood as model of *about a third*. Moreover, $\mathsf{W}_3(\Pi^1_1)$ might serve as a model of *very roughly half*, whereas $\mathsf{W}_2(\Pi^1_1)$ might be appropriate as fuzzy model of the (unhedged) determiner *half*.

In a similar manner, deliberate choice quantifiers can be used to generate plausible candidate models for the proportional reading of *many*. In particular, consider a model where asserting (the formal counterpart of) *Many* [*domain elements*] *are* ψ is expressed by a willingness to place a certain number of bets for random instances of $\psi(x)$. This amounts to considering the family of quantifiers Π^i_0. The corresponding truth functions are depicted in Figure 13.

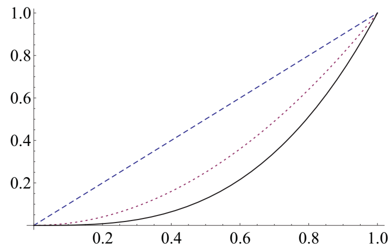


Figure 13. Truth functions for $\Pi_0^i x\psi(x)$ for $i = 1, 2, 3$ from top to bottom

Like for *about half* etc, above, one may want to evaluate *Many* [*domain elements*] *are* ψ as perfectly true (truth value 1) even if $\mathrm{Prop}_x\, \psi(x)$ is somewhat smaller than 1. Again, this can be achieved by employing the W_n-operator, which requires the attacker to place several bets against the contended assertion.

10 A brief synopsis of semantic games

To assist a comparison between the various semantic games that we have presented in Sections 2 to 8 we summarize their main characteristics in a table:

game	state determined by	payoffs	logic(s)	section
$\mathcal{H}$-game	single formula + role distribution	bivalent (win/lose)	CL	2
$\mathcal{H}$-mv-game	single formula + role distribution	many-valued	KZ	2
$\mathcal{E}$-game	single formula + role distribution + value	bivalent (win/lose)	Ł	3
$\mathcal{G}$-game(s)	two multisets (tenets) of formulas	many-valued (expected risk)	Ł, Ł_n, A CHL, ...	4, 5
$\mathcal{B}$-games	single formula + role distribution + stack	many-valued	Ł, G, Π	7
$\mathcal{R}$-game(s)	single formula + role distribution	many-valued (expected value)	$\mathrm{KZ}(\pi)$ $\mathrm{KZ}(\pi, \mathrm{D})$	8

11 Historical remarks and further reading

There are a lot of links between logic and games. We refer to [35] for an overview, including a brief history that goes back to antiquity. For our specific topic—*semantic games*, also known as *evaluation games*—Leon Henkin's [30] is an important precursor. Henkin pointed out that the game based understanding of universal and existential quantifiers may be applied to situations where Tarski's classic definition of truth in model fails. In the late 1960s Jaakko Hintikka [31] started to explore semantic games as an alternative to Tarski-style semantics for classical logic. Important references are [32] and the handbook chapter [33], written jointly with Gabriel Sandu.[11] This approach allowed to consider effects of incomplete information in semantic games under the name of IF (independence friendly) logic [41] (see also the hints on related topics, below). Independently, Rohit Parikh [48] also characterized classical and intuitionistic truth in terms of games. More importantly for our specific context, Robin Giles [25, 26], also already in the 1970s, suggested a game based interpretation of Łukasiewicz logic that is quite different form Hintikka's game, at least at a first glance. Giles seemingly was not aware of Hintikka's (or Parikh's) semantic games, but rather referred to the work of Paul Lorenzen and Kuno Lorenz as an inspiration. Lorenzen had suggested to model constructive validity by logical dialogue games already in the late 1950s [40]. This has then been taken up by Lorenzen's student Lorenz, e.g. in [39]. While Giles's rules for (weak) conjunction, (weak) disjunction, and the standard quantifiers are indeed close to Lorenzen's so-called particle rules for **P**-**O**-dialogues, Giles's game should be classified as a semantic game, since it characterizes (graded) truth with respect to a given interpretation. As explained in Section 4, for Giles an interpretation was specified by an assignment of probabilities to dispersive experiments associated with atomic formulas. However, at least with hindsight, it is clear that Giles's game implicitly refers to standard interpretations (i.e., interpretations over the real unit interval) of Łukasiewicz logic.

Cintula and Majer [9] generalized Hintikka's game to an 'evaluation game' for Łukasiewicz logic. We have called this game explicit evaluation game or $\mathcal{E}$-game, here, in order to distinguish it from other semantic games. In fact the rules presented here in Section 3 slightly deviate from those of [9], but are easily checked to be equivalent.

The 'smooth transition' from Hintikka's classical game to a many-valued setting presented in Section 2.2 is due to [21] and [22]. The limits for Hintikka-style games described in Section 2.3 have originally been presented in [17].

Presentations of Giles's game, that are close to that in Section 4 can be found, e.g., in [13, 22]. The connection between hypersequent systems and Giles-style dialogue games for t-norm based fuzzy logics has first been sketched in [8] and is explained in some more detail in [14]. Giles did not consider strong (t-norm) conjunction. Such a rule has been motivated and defined in [19].

The generalization(s) of Giles's game presented in Section 5 have largely been lifted from the paper [22].

As already pointed out in Section 6, the connection between Giles's game and a hypersequent system for Łukasiewicz logic has already been taken up in Chapter III of this Handbook [43] by George Metcalfe, based on the paper [19]. A related treatment can be found in [15]. The material on Abelian logic presented in Section 6 is new in principle. But, given the closeness of the respective hypersequents for Łukasiewicz logic and Abelian logic (see [44, 45]), it amounts to a straightforward exercise relative to the above mentioned sources. In fact, we have deliberately refrained from a more

[11] Hintikka uses *Myself* and *Nature* as names for the players and *Verifier* and *Falsifier* for the two roles. Sometimes the identity of the players is left implicit in semantic games and the Verifier is called *Eloise* or $\exists$*loise* or simply $\exists$, while the Falsifier is called *Abelard* or $\forall$*belard* or $\forall$. However, as already pointed out in Section 2, we refer to the players of all semantic games considered in this chapter as *myself (I)* and *you*, distinguishing between the role of an proponent **P** and an opponent **O**, respectively. This terminology can be traced back to Giles [25] and Lorenzen [40], respectively.

thorough formalization of the corresponding game along the lines of [19], here, in order to keep the presentation focused on essential features that emphasize the close relation to Giles's game for Łukasiewicz logic.

The backtracking games ($\mathcal{B}$-games) of Section 7 have originally been presented in [18]. The logic $G_{\sim}$, briefly mentioned at the end of Subsection 7.2, is considered at various places in the literature (see, e.g., [11]), but has not yet been considered from a game semantic point of view.

Propositional random choice games ($\mathcal{R}$-games), that are the topic of Section 8, are due to [16] (where a synopsis similar to that presented here as Section 10 can be found).

The presentation of random choice rules for semi-fuzzy quantifiers in Section 9 closely follows that of [22]. The idea of using random choices of witness constants for modeling the evaluation of formulas involving semi-fuzzy quantifiers in the context of Łukasiewicz logic has first been presented in [21].

We finally mention some related topics that have not been covered in this chapter:

- As already mentioned above, Lorenzen [40] defined a dialogue game that was intended to characterize intuitionistic validity. In [12] it is shown how parallel dialogue games that model different synchronization mechanisms between underlying Lorenzen-style dialogues result in characterizations of various intermediate logics, among them Gödel logic. These parallel dialogue games are closely related to hypersequent systems and in particular lead to a game based interpretation of Avron's so-called communication rule [2].
- A game for Gödel logic that could be classified as a semantic game is implicitly presented in [20]. It proceeds by reducing claims about the relative order of truth degrees of complex formulas to claims only involving atomic formulas. Evaluating the resulting claims with respect to a given interpretation yields a semantic game. However the intended use of the game in [20] was to check the validity of formulas. For this purpose the game continues after reduction to atomic order claim in such a way that the opponent **O** wins the game if the initial formula is not valid. The game is related to the sequent-of-relations system introduced in [4].
- We have not dealt with finitely-valued logics in a systematic manner here. In fact, it is rather straightforward to describe explicit evaluation games for all logics that are specified by finite truth tables (matrices). As explained in [17], this in turn can be generalized to all logics characterized by so-called Nmatrices (nondeterministic) matrices, introduced in [3].
- A particular interesting topic that has not yet been fully explored is the relation between the game based equilibrium semantics for IF logic (independence friendly logic, see [41, 52]) and many-valued logics. IF logic results from extending Hintikka's (perfect information) game for classical logic to incomplete information, i.e., to situations where at some states not all previous moves are known to both players. The truth functions $\min$, $\max$, and $1 - x$ over $[0, 1]$ for conjunction, disjunction, and negation, respectively, naturally arise in this context if we look for Nash equilibria in such games. As pointed out, e.g., in [16, 54], further propositional connectives, including the operator π, treated in Section 8, can be modeled by semantic games of incomplete information.

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